PHILOSOPHY 511:
SYMBOLIC LOGIC

PETER B. M. VRANAS
MWF 11-11:50, SPRING 2007

QUIZ #1

Please indicate on the answer sheet whether the following four statements are true or false

1. Assuming that the identity sign is present, the empty language has infinitely many formulas.

2. The identity symbol is a nonlogical symbol.

3. When formulas are written in official notation, every formula begins with a left parenthesis.

4. In '$(\forall x(R(y)\ \&\ Q(z))\ \&\ P(x, y))$', a formula written in official notation, the first occurrence of 'x' is bound and the second occurrence is free.

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QUIZ #2

Please indicate on the answer sheet whether the following four statements are true or false

1. In every interpretation of a language, every element of the domain is the denotation of some closed term.

2. When identity is present, the sentence '$\exists x(x=x)$' is true in every interpretation of a language.

3. In the standard interpretation $\mathscr{N}^*$ of the language of arithmetic L^*, the sentence '$\exists x(x+x=1)$' is true.

4. According to BBJ, the right approach to defining truth in the case of quantification is the *substitutional*, not the *objectual*, approach.

QUIZ #3

Please indicate on the answer sheet whether the following four statements are true or false

1. *Inconsistency* was defined as a *semantic*, not a *syntactic*, notion.

2. A sentence was defined to be *demonstrable* exactly if no interpretation makes it false.

3. A proof procedure was defined to be *sound* exactly if every secure sequent is derivable according to the procedure.

4. A metalanguage for an object language is always different from the object language.

QUIZ #4

Please respond on the answer sheet to 1 and 2, and indicate on the answer sheet whether statements 3 and 4 are true or false

1. Formulate, *in five words,* the Gödel completeness theorem.

2. A *refutation* of Γ is a derivation of which sequent?

3. It is possible for a proof procedure to be complete but not sound.

4. The following is a correct application of rule (R6)—*left existential quantifier introduction*:

$$\frac{t{=}t \;\&\; B(t) \Rightarrow}{\exists x(x{=}t \;\&\; B(x)) \Rightarrow}$$

QUIZ #5

Please indicate on the answer sheet whether the following four statements are true or false

1. If some finite subset of a set of sentences is unsatisfiable, then the whole set of sentences is unsatisfiable.

2. If a set of sentences is unsatisfiable, then some proper subset of this set is also unsatisfiable.

3. If a set of sentences is satisfiable, then it has a model whose domain is the set of natural numbers.

4. If a set of sentences has an enumerable model, then it has a finite model.

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QUIZ #6

Please indicate on the answer sheet whether the following four statements are true or false

1. Every axiomatizable theory is decidable.

2. The set $\{A, \sim A, A \rightarrow A, A \rightarrow \sim A, \sim A \rightarrow A, \sim A \rightarrow \sim A\}$ is a theory.

3. If there is a sentence not contained in a theory, then the theory is consistent.

4. A sentence is *undecidable* in a theory exactly if there is no effective procedure for determining whether or not the sentence is contained in the theory.

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QUIZ #7

Please indicate on the answer sheet whether the following four statements are true or false

1. In every system of Gödel numbering, every natural number is the Gödel number of some symbol.

2. In George and Velleman's system of Gödel numbering, given any formula, there is some symbol such that the Gödel number of the formula is the same as the Gödel number of the symbol.

3. Every representable set is decidable, but not vice versa.

4. The set of Gödel numbers of sentences of the language of PA is representable.

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QUIZ #8

Please indicate on the answer sheet whether the following four statements are true or false

1. It is a consequence of the fixed point lemma that there is a sentence Q such that: PA $\vdash Q \leftrightarrow (\ulcorner Q \urcorner = 0)$.

2. Every consistent theory is also ω-consistent.

3. There is an algorithm that determines, for any given sentence of the language of PA, whether or not that sentence is true.

4. If T is a consistent extension of PA, then the set of Gödel numbers of theorems of T is not representable.

QUIZ #9

Please indicate on the answer sheet whether the following four statements are true or false

1. According to Gödel's Second Incompleteness Theorem, no consistent extension of PA can prove its own consistency.

2. It is a consequence of Gödel's Second Incompleteness Theorem that the consistency of PA can only be proven in a theory *stronger* than PA.

3. If a theory T is an axiomatizable extension of PA, then PA contains the sentence '$Con_T \leftrightarrow G_T$', where Con_T is the consistency sentence of T and G_T is the Gödel sentence of T.

4. A theory which is an extension of PA is consistent exactly if it does not contain the sentence '$0{=}S0$'.

QUIZ #10

Name: ______________________________

Please write to the left of each number either 'TRUE' or 'FALSE'

1. According to George and Velleman, on Hilbert's view finitary truths obtain in virtue of correctly describing some independently existing realm of abstract entities.

2. According to George and Velleman, on Hilbert's view all existential quantifications are meaningless.

3. According to George and Velleman, Hilbert's project aims at assigning intuitionistically intelligible meanings to the statements of infinitary mathematics.

4. According to George and Velleman, if we can give a finitary proof that an axiomatic theory is consistent, then the theory is a conservative extension of finitary mathematics.

OPTIONAL FEEDBACK QUESTIONS

Please evaluate the reading for today:

□ very good □ good □ fair □ poor □ very poor

Comments on the reading: ______________________________

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QUIZ #11

Name: ______________________________

Please write to the left of each number either 'TRUE' or 'FALSE'

1. According to George and Velleman, the belief that one can capture within one formal system all the basic concepts and forms of reasoning of infinitary mathematics is only peripheral to Hilbert's project.

2. According to George and Velleman, the point of Hilbert's project is that, if infinitary formal mathematics solves a problem, then that solution can be trusted to be correct.

3. According to George and Velleman, for any formal system, the negation of its Gödel sentence is intuitionistically true but finitarily meaningless.

4. According to George and Velleman, the phenomenon of incompleteness opens the possibility that a formal system is proved consistent by finitary means even though one can derive within it a statement that is intuitionistically false.

OPTIONAL FEEDBACK QUESTIONS

Please evaluate the reading for today:

□ very good □ good □ fair □ poor □ very poor

Comments on the reading: ______________________________

Please evaluate the last lecture:

□ very good □ good □ fair □ poor □ very poor

Comments on the lecture: ______________________________

QUIZ #12

Name: ______________________________

Please write to the left of each number either 'TRUE' or 'FALSE'

1. According to George and Velleman, Gödel's Second Incompleteness Theorem guarantees the unrealizability of Hilbert's project.

2. According to George and Velleman, Hilbert was not completely explicit regarding the scope and resources of finitary reasoning.

3. According to George and Velleman, Gödel's work forces one to accept a realist construal of mathematics.

4. According to George and Velleman, intuitionists would infer from Gödel's theorems that the concept of being intuitionistially provable transcends that of being provable in PA.

OPTIONAL FEEDBACK QUESTIONS

Please evaluate the reading for today:

□ very good □ good □ fair □ poor □ very poor

Comments on the reading: ______________________________

Please evaluate the last lecture:

□ very good □ good □ fair □ poor □ very poor

Comments on the lecture: ______________________________

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QUIZ #13

Name: ______________________________________

Please write to the left of each number either 'TRUE' or 'FALSE'

1. According to Franzén, we know that ZFC is consistent because we know that its axioms are true.

2. According to Franzén, if we have no doubts about the consistency of ZFC, there is nothing in the Second Incompleteness Theorem to give rise to any such doubts.

3. According to DeLong, there is a deterministic proof of the consistency of elementary arithmetic.

4. According to DeLong, even if an inconsistency in arithmetic should turn up, it might be unimportant.

OPTIONAL FEEDBACK QUESTIONS

Please evaluate the reading for today:

□ very good □ good □ fair □ poor □ very poor

Comments on the reading: ______________________________________

Please evaluate the last lecture:

□ very good □ good □ fair □ poor □ very poor

Comments on the lecture: ______________________________________

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QUIZ #14

Name: ______________________________

Please write to the left of each number either 'TRUE' or 'FALSE'

1. According to Franzén, Gödel's proof of the First Incompleteness Theorem shows that the Gödel sentence of any consistent and sufficiently powerful formal system is true (in the standard interpretation).

2. According to Franzén, Lucas has argued that, for any consistent formal system *S*, there is a true statement that we can prove but *S* cannot.

3. According to Franzén, we have no basis for claiming that we ("the humand mind") can outprove any given consistent formal system.

4. According to Franzén, the First Incompleteness Theorem rules out that every arithmetical statement provable in ZFC can also be proved by the human mind.

OPTIONAL FEEDBACK QUESTIONS

Please evaluate the reading for today:

□ very good □ good □ fair □ poor □ very poor

Comments on the reading: ______________________________

Please evaluate the last lecture:

□ very good □ good □ fair □ poor □ very poor

Comments on the lecture: ______________________________

QUIZ #15

Name: ______________________________

Please write to the left of each number either 'TRUE' or 'FALSE'

1. According to Lucas, if the game played between the mentalist and the mechanist is the game of one-upmanship, the result must be a draw.

2. According to Lucas, for any particular person, no specified machine is equal, in respect of intellectual powers, to that particular person.

3. According to Lewis, Lucas is really claiming to be able to do something that no machine could do.

4. According to Lewis, there are no machines that respond to true mechanistic accusations by producing true Gödel sentences.

OPTIONAL FEEDBACK QUESTIONS

Please evaluate the reading for today:

□ very good □ good □ fair □ poor □ very poor

Comments on the reading: ______________________________

Please evaluate the last lecture:

□ very good □ good □ fair □ poor □ very poor

Comments on the lecture: ______________________________

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QUIZ #16

Name: ______________________________

Please write to the left of each number either 'TRUE' or 'FALSE'

1. According to Gaifman, McCall observes that it is unknown whether ω-consistency can be replaced by consistency in the second part of Gödel's Theorem.

2. According to Gaifman, if I believe that my mathematical reasoning is immune to contradiction, and if the formal system T represents faithfully my mathematical reasoning, then I should believe that T is consistent.

3. According to Gaifman, in order to believe the consistency of some program on *mathematical* grounds, we have to understand how the program works.

4. According to Gaifman, Gödel's result shows that self-reflection cannot encompass the whole of our reasoning.

OPTIONAL FEEDBACK QUESTIONS

Please evaluate the reading for today:

□ very good □ good □ fair □ poor □ very poor

Comments on the reading: ______________________________

Please evaluate the last lecture:

□ very good □ good □ fair □ poor □ very poor

Comments on the lecture: ______________________________

QUIZ #17

Name: ______________________________

Please write to the left of each number either 'TRUE' or 'FALSE'

1. According to Franzén, (his proposed reconstruction of) "Penrose's second argument" is "correct" (i.e., valid).

2. According to Franzén, we cannot specify any formal system that exhausts our mathematical knowledge.

3. According to Franzén, Penrose claims that his conclusion *G* is incompatible with the assumption that the human mind is exactly equivalent to some formal system as far as its ability to prove arithmetical statements is concerned.

4. According to Franzén, the claim that no "knowably sound" algorithm or theory exhausts our arithmetical knowledge is based on a misunderstanding of the proof of the First Incompleteness Theorem.

OPTIONAL FEEDBACK QUESTIONS

Please evaluate the reading for today:

□ very good □ good □ fair □ poor □ very poor

Comments on the reading: ______________________________

Please evaluate the last lecture:

□ very good □ good □ fair □ poor □ very poor

Comments on the lecture: ______________________________

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QUIZ #18

Name: ______________________________

Please write to the left of each number either 'TRUE' or 'FALSE'

1. According to Franzén, Gödel's First Incompleteness Theorem does not apply to the philosophy of Ayn Rand.

2. According to Franzén, the question of the actual or potential reach of the human mind when it comes to proving theorems in arithmetic is *not* like the question of how many hot dogs a human can eat in five minutes without making a totally disgusting spectacle of himself.

3. According to Franzén, it seems reasonable to assume that a formalization of theoretical physics, if such a theory can be produced, would be subject to the incompleteness theorem.

4. It is unknown whether the following is true: given any even number (greater than 2) of (wooden) squares of equal size, one can always divide them into two piles, each of which cannot be arranged in a rectangle each side of which has at least two squares.

OPTIONAL FEEDBACK QUESTIONS

Please evaluate the reading for today:

□ very good □ good □ fair □ poor □ very poor

Comments on the reading: ______________________________

Please evaluate the last lecture:

□ very good □ good □ fair □ poor □ very poor

Comments on the lecture: ______________________________

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THE UNIVERSITY WISCONSIN MADISON

PETER B. M. VRANAS
MWF 11-11:50, SPRING 2007

PROBLEM SETS

PROBLEM SET #1

Due: Monday 29 January at 11:00 am in class

Problem 1 (1 point). For each of the following eight strings of symbols, write in the corresponding space of the answer sheet: 'A' if the string is not a term, 'B' if the string is an atomic closed term, 'C' if the string is an atomic open term, 'D' if the string is a non-atomic closed term, and 'E' if the string is a non-atomic open term (*in the language of arithmetic* L^**, using unofficial notation*).

(a) $P(x, y)$	(e) $x \neq y$
(b) $2+3$	(f) z
(c) $0+x$	(g) $2<3$
(d) 458	(h) $x'+y'''$

Problem 2 (1 point). For each of the following eight strings of symbols, write in the corresponding space of the answer sheet: 'A' if the string is not a formula, 'B' if the string is an atomic formula and a sentence, 'C' if the string is an atomic formula but not a sentence, 'D' if the string is a non-atomic formula and a sentence, and 'E' if the string is a non-atomic formula but not a sentence (*in the language of arithmetic* L^**, using unofficial notation*).

(a) $\forall x(x \neq x \,\&\, x=3)$	(e) $\forall x \exists y \forall z \exists w(x<y \,\&\, z<x)$
(b) $\forall x \forall y(x>y \rightarrow y<x)$	(f) $2+3\bullet(2\bullet x+3\bullet \mathrm{x})<y+(3\bullet x+z)$
(c) $0+x<1+x$	(g) $\forall x \exists x(x=x)$
(d) $454=455$	(h) $\exists z(2<x \,\&\, \forall z\, x<z)$

Problem 3 (1 point). For each of the following four formulas (in the language $L^* \cup \{P, Q, R\}$, where P and Q are two-place predicates and R is a one-place predicate; *official notation is used*), write in the corresponding space of the answer sheet all free occurrences of variables in the following format: ${}^2x$ if the second occurrence of 'x' is free in the formula, and so on.

(a) $(P(x, x) \,\&\, \forall x Q(x, x))$	(c) $\forall x \exists y(P(x, y) \,\&\, R(y))$
(b) $\forall x(\exists y P(x, y) \,\&\, R(y))$	(d) $((\sim Q(2, 3) \,\&\, \forall z Q(z, 3)) \vee \exists y(P(z, y) \rightarrow \forall z \exists x R(x)))$

Problem 4 (1 point). Do problem 9.3 on p. 113 of BBJ.

PROBLEM SET # 2

Due: Monday 5 February at 11:00 am in class

Problem 1 (1 point). Do Problem 9.2 on p. 112 of BBJ.

Problem 2 (1 point). Do exercise D on p. 239 of Bessie & Glennan (see next pages), *only* #71.

Problem 3 (1 point). Do exercise C on pp. 237-239 of Bessie & Glennan (see next pages), *only* #45, 47, 48, 50, 51, 54, 57, 58.

Problem 4 (1 point). Translate into logical notation the following argument, using the notation provided. (1) A *limit point* of a set is the limit of a sequence all terms of which are members of the set. (2) A set is *closed* exactly if every limit point of the set is a member of the set. (3) A *collective* is a set all members of which are sets. (4) The *intersection* of a collective is a set whose members are all and only the members of *every* member of the collective. It follows from the above four *definitions* that (5) the intersection of a collective all members of which are closed is closed. (*Cx*: *x* is closed; *Kx*: *x* is a collective; *Sx*: *x* is a set; *Qx*: *x* is a sequence; *Ixy*: *x* is the intersection of *y*; *Lxy*: *x* is the limit of *y*; *Mxy*: *x* is a member of *y*; *Pxy*: *x* is a limit point of *y*; *Txy*: *x* is a term of *y*.)

FROM: BESSIE & GLENNAN, ELEMENTS OF DEDUCTIVE INFERENCE

Exercises for Section 5.5

(A) Below are listed three interpretations, followed by several statements involving monadic predicates. Evaluate the truth values of each statement, using $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$.

$\mathcal{I}_1$ $\mathcal{D}$: animals
Ax: x is a mammal
Bx: x is a ferret
Cx: x is female
a: Mickey Mouse
b: Minnie Mouse
c: Donald Duck

$\mathcal{I}_2$ $\mathcal{D}$: natural numbers {1, 2, 3, . . .}
Ax: x is even
Bx: x is odd
Cx: x is prime
a: 2
b: 3
c: 4

$\mathcal{I}_3$ $\mathcal{D}$: {Burt, Ernie, Elmo, Cookie Monster, Big Bird}
Ax: {Burt, Ernie}
Bx: {Burt, Elmo, Cookie Monster}
Cx: ∅
a: Big Bird
b: Elmo
c: Cookie Monster

1. Ca
2. ~(Aa → Bb)
♦3. ~(Ba ∨ Ca)
4. ~Ba ∨ Ca
5. ∃xAx
6. ~∀x~Cx
♦7. ∃xAx & ∃x~Bx
8. ∀x(Bx ↔ Ax)
9. ∃x(Ax & Cx)
10. ∃x(Cx ∨ ~Cx)
11. ∀x(Bx → Cx)
12. ∃x(Bx → Cx)
♦13. ∀xBx → Ca
14. (Ba & Bb) → ∀zBz
15. ∀x[(Ax & Bx) → ~Cx]

(B) For each of the following *L* statements involving monadic predicates, provide two interpretations: one under which the given statement is true and one under which the given statement is false.

Example: Given: ∀x(Ax → Bx)

1. $\mathcal{D}$:	people		2. $\mathcal{D}$:	people
Ax:	x is a person		Ax:	x is a man
Bx:	x has a mother		Bx:	x is a father

Under interpretation 1 the statement says 'All people have mothers', which is true. Under interpretation 2 the statement says 'All men are fathers', which is false.

16. Ar
17. ∀xAx → ∃yBy
18. Br & ∀xAx
♦19. Ar & Br
20. ∃xAx → ∀yBy
21. Mg → ∀x(Bx ↔ Mx)
22. Ma ∨ Fb
23. ∃xAx → ∀xAx
24. (∃xAx & ∃xBx) → ∀zKz
♦25. ∃x(Fx ∨ Mx)
26. ∃zFz ∨ ∃yGy
27. ∀x(Ax → Bx) → ∃x~Cx
28. ∃x(Fx & Mx)
29. ~∃x(Ax ∨ Bx)
30. ∀x((Ax & Bx) ↔ Cx)
♦31. ∃xAx → ∀xBx
32. ~∀x(Ax → Bx)
33. ∀x[Fx → (Gx ∨ Hx)]
34. ∀y[(Ay & By) & Cy]
35. ∀xWx → ∀z(Vz ∨ ~Pz)

PROBLEM 3 OF PROBLEM SET #2

(C) Below are listed three interpretations, followed by several statements involving polyadic predicates. Evaluate the truth values of each statement, using $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$.

$\mathcal{I}_1$ $\mathcal{D}$: natural numbers {1, 2, 3, . . .}
Gxy: x is greater than y
Lxy: x is less than y
Sxyz: the sum of x and y is z
Pxyz: the product of x and y is z
Ex: x is even
Ox: x is odd
a: 1
b: 2
c: 3

$\mathcal{I}_2$

𝒟:	{Susan, Emily, Lucinda, Samantha, Kirk, Lily, Holden, James, David, Molly, Ben, Camille, Cal, Bob, Kim, Jack, Margo, Tom, Adam}
Gxy:	{<Lucinda, Lily>, <Kim, Tom>, <Susan, Emily>, <Margo, Adam>}
Lxy:	{<Tom, Adam>, <Bob, Tom>}
Sxyz:	{<Tom, Margo, Adam>, <Margo, Tom, Adam>, <Kim, Bob, Tom>, <Bob, Kim, Tom>}
Pxyz:	{<Lucinda, Samantha, Lily>, <Samantha, Lucinda, Lily>, <Lily, Lucinda, Samantha>, <Samantha, Lily, Lucinda>}
Ex:	{Holden, James, David, Ben, Cal, Bob, Tom, Jack}
Ox:	{Susan, Emily, Lucinda, Samantha, Lily, Molly, Camille, Kim, Margo}
a:	Lucinda
b:	Samantha
c:	Lily

$\mathcal{I}_3$

𝒟:	states of the USA
Gxy:	x is larger in area than y
Lxy:	x is more populous than y
Sxyz:	x is is between y and z (ie., east of y and west of z)
Pxyz:	x is bordered by y and z
Ex:	x is a coastal state
Ox:	x is one of the contiguous 48 states
a:	Alaska
b:	Maryland
c:	Illinois

36. Sabc
37. Pabb
38. Gca & Lac
♦39. Lab & ~Gac
40. ∃x(Ex & Gxb)
41. ∃x(Ox & Gxc)
42. ~∃xGxx
43. ∃xSabx
44. ∃x(Sccx & Ox)
♦45. ∀x∀y(Gxy → ~Gyx)
46. ∀x∀y(Gxy → Lyx)
47. ∀x∀y(~Gxy → Lxy)
48. ∀x∀y∃zSxyz
49. ∀x∃yGxy
50. ∃x∀yGxy
51. ∀x∀y(Gxy → ∃zSyzx)
52. ∀x∀y(Gxy → ∃zPyzx)
♦53. ∀x∀y(Lxy → ∃zSxzy)
54. ∀x∀y∀z(Pxyz → ∃uSxuz)

55. ∀x∀y((Ex & Oy) → ∃z(Sxyz & Oz))
56. ∀x∀y((Ex & Oy) → ∃z(Sxyz & Ez))
57. ∀x∀y((Ex & Oy) → ∃z(Pxyz & Oz))
58. ∀x∀y((Ex & Oy) → ∃z(Pxyz & Ez))

PROBLEM 2 OF PROBLEM SET #2

(D) For each of the following *L* statements involving polyadic predicates, provide two interpretations: one under which the given statement is true and one under which the given statement is false.

Example: Given: ∀x∀y(Axy → Bxy)

1. 𝒟:	people	2. 𝒟:	people
Axy:	x is a parent of y	Axy:	x is the sister of y
Bxy:	y is a child of x	Bxy:	y is the brother of x

Under interpretation 1. the statement says 'Anybody who is the parent of another has that other as their child', which is true. Under interpretation 2. the statement says 'Anybody who is the sister of another has that other person as her brother', which is false.

59. Mab
60. ~Mab
61. Mab ∨ Mba
62. Mab & Mba
♦63. ∀xMax
64. ∃xMax & ~∀xMax
65. ∀x∀y(Lxy → Lyx)
66. ∀x∀y(Lxy → ~Lyx)
67. ∀x∃yAxy
68. ∃x∀yAxy
♦69. ∀x∀yAxy
70. ∃x∃yAxy & ∃z∃u~Azu
71. ∀x(Pxa → ∃yQxy)
72. ∃xAxa ∨ ∀yBay
73. Rabc
74. ∃xRabx
75. ∃xRaxb
76. ∃xRxbc

5.6 Symbolizing English I: Monadic Logic and Categorical Forms

Because *L* enables us to symbolize aspects of the logical structure of English statements that exist within atomic statements, *L* is a much more powerful tool for the analysis of arguments than is *SL*. The task of symbolizing English statements in *L* is, however, correspondingly more complex. We divide our discussion of symbolization into two sections. In this section we concentrate on monadic predicates. In Section 5.7 we turn to polyadic predicates and multiple quantifiers.

PROBLEM SET # 3

Due: Monday 12 February at 11:00 am in class

Problem 1 (1 point). Do problem 10.4 on p. 123 of BBJ.

Problem 2 (1 point). Do problem 10.6 on p. 124 of BBJ.

Problem 3 (1 point). Do problem 10.12 on p. 124 of BBJ.

Problem 4 (1 point). Do problem 14.1(a)—*not* (b)—on p. 185 of BBJ.

PROBLEM SET # 4

Due: Monday 19 February at 11:00 am in class

Problem 1 (1 point). Provide an example of an improper use of the second quantifier rule, namely (R6), arising from ignoring the side condition 'c not in $A(x)$' (see p. 173 of BBJ, after Example 14.10).

Problem 2 (1 point). Provide a derivation (using (R0)-(R9)) of:
$\exists y \forall x R(x, y) \Rightarrow \forall x \exists y R(x, y)$.

Problem 3 (1 point). Provide a derivation (using (R0)-(R9)) of:
$\exists x F(x)$ & $\exists y G(y) \Rightarrow \exists x \exists y (F(x)$ & $G(y))$.
Start the derivation with '$F(a), G(b) \Rightarrow F(a)$ & $G(b)$', giving as justification 'class discussion'.

Problem 4 (1 point). (a) Provide an example of a proof procedure that is sound but not complete. Explain why it is sound but not complete. (b) Provide an example of a proof procedure that is complete but not sound. Explain why it is complete but not sound.

PROBLEM SET # 5

Due: Monday 5 March at 11:00 am in class

Problem 1 (2 points). Suppose Γ_1 and Γ_2 are sets of sentences such that $\Gamma_1 \cup \Gamma_2$ is unsatisfiable. Show that there is a sentence S such that $\Gamma_1 \models S$ and $\Gamma_2 \models \sim S$. (Hint: Use the compactness theorem.)

Problem 2 (2 points). Give either a proof of, or a counterexample to, the following statement: If for every two members γ_1, γ_2 of a set of sentences Γ the set $\{\gamma_1, \gamma_2\}$ is satisfiable, then Γ is satisfiable.

PROBLEM SET # 6

Due: Monday 12 March at 11:00 am in class

Problem 1 (2 points). A certain island is inhabited only by knights, who always tell the truth, and knaves, who always lie. Furthermore, some of the knights have proven themselves to be knights, and are known as *established knights*, and similarly some of the knaves are *established knaves.*

(a) An inhabitant of the island says 'I am not an established knight'. Can you tell whether he is a knight or a knave, and whether or not he is established?
(b) An inhabitant of the island says 'I am an established knave'. Can you tell whether he is a knight or a knave, and whether or not he is established?
(c) An inhabitant of the island says 'I am an established knight'. Can you tell whether he is a knight or a knave, and whether or not he is established?

Problem 2 (2 points). Solve the problem in 'Machines that talk about themselves' (see next pages).

PROBLEM 2 OF PROBLEM SET #6

Machines That Talk About Themselves

We shall now consider Gödel's argument from a slightly different perspective, which puts the central idea in a remarkably clear light.

We shall take the four symbols P,N,A, – and consider all possible combinations of these symbols. By an *expression* we mean any combination of the symbols. For example, P – – NA – P is an expression; so is – PN – – A – P – . Certain expressions will be assigned a meaning, and these expressions will be called *sentences.*

Suppose we have a machine that can print out some expressions but not others. We call an expression *printable* if the machine can print it. We assume that any expression that the machine can print will be printed sooner or later. Given any expression X, if we wish to express the proposition that X is printable, we write P – X. So, for example, P – ANN says that ANN is printable (this may be true or false, but that's what it says!). If we want to say that X is *not* printable, we write NP – X. (The symbol N is the abbreviation of the word *not,* just as the symbol P represents the word *printable.* And so NP – X is to be read, crudely, as "Not printable X," or, in better English, "X is not printable.")

By the *associate* of an expression X we mean the expression X – X. We use the symbol A to stand for "the associate of," and so, for any given X, if we wish to state that the associate of X is printable we write PA – X (read "printable the associate of X," or in better English, "the associate of X is printable"). If we wish to say that the associate of X is *not* printable, we write NPA – X (read "not printable the associate of X," or, in better English, "the associate of X is not printable").

Now, the reader may well wonder why we use the dash as a symbol: why don't we simply use PX rather than P – X to express the proposition that X is printable? The reason is that omission of the dash would create a contextual ambiguity. What, for example, would PAN mean? Would it mean that the associate of N is printable or that the expression AN is printable? With the use of the dash, no such ambiguity arises. If we want to say that the associate of N is printable, we write down PA – N; whereas, if we want to say that AN is printable, we write down P – AN. Again, suppose we want to say that – X is printable; do we write P – X? No, that would state that X is printable. To say that – X is printable, we must write P – – X.

Perhaps some more examples might help: P – – says that – is printable; PA – – says that – – – (the associate of –) is printable; P – – – – also says that – – – is printable; NPA – – P – A says that the associate of – P – A is not printable; in other words, that – P – A – – P – A is not printable. NP – – P – A – – P – A says the same thing.

We now define a *sentence* as any expression of one of the four forms P – X, NP – X, PA – X, and NPA – X, where X is any expression whatever. We call P – X *true* if X is printable, and *false* if X is not printable. We call NP – X true if X is *not* printable and false if X *is* printable. We call PA – X true if the associate of X is printable, and false if the associate of X is not printable. Finally, we call NPA – X true if the associate of X is not printable, and false if the associate of X is printable.

We have now given a precise definition of truth and falsity for sentences of all four types, and from this it follows that, for any expression X:

Law 1: P – X is true if and only if X is printable (by the machine).
Law 2: PA – X is true if and only if X – X is printable.
Law 3: NP – X is true if and only if X is not printable.
Law 4: NPA – X is true if and only if X – X is not printable.

We have here a curious loop! The machine is printing out sentences that make assertions about what the machine can and cannot print! In this sense, the machine is talking about itself (or, more accurately, printing out sentences about itself).

We are now given that the machine is a hundred percent accurate—that is, it never prints out any false sentence; it prints out only true sentences. This fact has several ramifications: As an example, if it ever prints out P – X, then it must also print out X, because, since it prints out P – X, then P – X must be true, which means that X is printable, and hence the machine must sooner or later print X.

It follows as well that if the machine should print out PA – X, then (since PA – X must be true), the machine must also print out X – X. In addition, if the machine prints out NP – X, then it *cannot* also print P – X, since these two sentences can't both be true—the first says that the machine doesn't print X, and the second says that the machine does print X.

The following problem puts Gödel's idea into as clear a light as any problem I can imagine.

A Singularly Gödelian Challenge

Find a true sentence that the machine cannot print!

PROBLEM SET # 7

Due: Monday 19 March at 11:00 am in class

Problem 1 (1 point). Find (as a product of prime numbers) the Gödel number of the sentence '$\forall x_1((x_1 \bullet 0)=0)$', using George and Velleman's system of Gödel numbering.

Problem 2 (1 point). Find the expression whose Gödel number (in George and Velleman's system of Gödel numbering) is:
$8{,}000 \bullet 5{,}764{,}801 \bullet 43{,}046{,}721 \bullet 11^{12} \bullet 13^{16} \bullet 17^{10} \bullet 19^{11} \bullet 23^{9}$.

Problem 3 (2 points). Following the instructions after (6) on p. 181 of George and Velleman, and letting the two formulas that correspond to Theorem 7.9 on p. 180 be P_i and P_{ii}, turn (6) into a formula of the language of PA.

PROBLEM SET # 8

Due: Monday 26 March at 11:00 am in class

Problem 1 (2 points). Let T be the theory whose axioms are those of PA together with the sentence $\sim G_{PA}$. Assume that PA is consistent. (a) Show that T is consistent. (b) Show that T is ω-inconsistent.

Problem 2 (2 points). Solve the problem in 'Fergusson's logic machine' (see next pages).

PROBLEM SET # 9

Due: Monday 9 April at 11:00 am in class

Problem 1 (2 points). Define a logician to be *accurate* if everything she can prove is true; she never proves anything that it is false. One day, an accurate logician visited the island of knights and knaves, in which each inhabitant is either a knight or a knave, and knights make only true statements and knaves make only false ones. The logician met a native who made a statement from which it follows that the native must be a knight, but the logician can never prove that he is! What statement would work?

Problem 2 (2 points). (To be read *after* solving the previous problem.) Suppose we are given the additional information that the logician can do logic at least as well as you and I. In the solution to the last problem, one proves that the native must be a knight. What is to prevent the logician from going through the same reasoning and hence coming up with the conclusion that the native is a knight? She would thus *prove* that the native is a knight, which would falsify the native's statement, so the native would be a knave! How are we to avoid this paradox?

PROBLEM 2 OF PROBLEM SET #8

matter to decide purely mechanically whether a given sequence of sentences is or is not a proof in the system; indeed, it is a simple matter to construct a machine that does this. It is an altogether different matter to construct a machine that will decide which sentences of an axiom system are provable and which ones are not. Whether or not this can be done may, I suspect, depend on the axiom system. . . .

"My current interest is in mechanical theorem-proving—that is, in machines that prove various mathematical truths. Here is my latest one," Fergusson said, pointing proudly to an extremely odd-looking contraption.

Craig and McCulloch stood several minutes before the machine trying to figure out its functions.

"Just what does it do?" Craig finally asked.

"It proves various facts about the positive whole numbers," replied Fergusson. "I am working in a language that contains names of various sets of numbers—specifically, positive integers. There are infinitely many sets of numbers nameable in this language. For example, we have a name for the set of even numbers, one for the set of odd numbers, one for the set of prime numbers, one for the set of all numbers divisible by 3—just about every set that number-theorists are interested in has a name in the language. Now, although there are infinitely many nameable sets, there are no more nameable sets than there are positive integers. And to each positive integer n is associated a certain nameable set A_n. We can thus think of all the nameable sets arranged in an infinite sequence $A_1, A_2, \ldots, A_n, \ldots$ (If you like, you can think of a book with infinitely many pages, and for each positive integer n, the nth page contains a description of a set of positive integers. Then think of the set A_n as the set described on page n of the book.)

"I employ the mathematical symbol 'ϵ,' which represents

the English phrase 'belongs to' or 'is a member of,' and for every number x and every number y, we have the sentence $x \epsilon A_y$, which is read 'x belongs to the set A_y.' This is the only type of sentence my machine investigates; the function of the machine is to try and discover what numbers belong to what nameable sets.

"Now, each sentence $x \epsilon A_y$ has a *code* number—namely, a number which, when written in the usual base 10 notation, consists of a string of 1's of length x followed by a string of 0's of length y. For example, the code number of the sentence $3 \epsilon A_2$ is 11100; the code number of $1 \epsilon A_5$ is 100000. For any x and y, by $x{*}y$ I mean the code number of the sentence $x \epsilon A_y$; thus, $x{*}y$ consists of a string of 1's of length x followed by a string of 0's of length y.

"The machine operates in the following manner," continued Fergusson. "Whenever it discovers that a number x belongs to a set A_y, it then prints out the number $x{*}y$—the code number of the sentence $x \epsilon A_y$. If the machine prints $x{*}y$, then I say that the machine has *proved* the sentence $x \epsilon A_y$. And I say that the sentence $x \epsilon A_y$ is *provable* (by the machine) if the machine is capable of printing out the number $x{*}y$.

"Now, I know that my machine is always accurate in the sense that every sentence provable by the machine is true."

"Just a moment," interrupted Craig, "what do you mean by *true*? How does *true* differ from *provable*?"

"Oh," replied Fergusson, "the two concepts are entirely different: I call a sentence $x \epsilon A_y$ *true* if x is really a member of the set A_y. That is entirely different from saying that the machine is capable of printing out the number $x{*}y$. If the latter holds, then I say that the sentence $x \epsilon A_y$ is *provable*—that is, by the machine."

"Oh, now I understand," said Craig. "In other words, when you say that your machine is accurate—that every sentence provable by the machine is a true sentence—what you mean is that the machine never prints out a number $x{*}y$ unless x is really a member of the set A_y. Is that correct?"

"Exactly!" replied Fergusson.

"Tell me," said Craig, "how do you know that your machine is always accurate?"

"To answer that," replied Fergusson, "I must tell you all the details of the machine. The machine operates on the basis of certain axioms about the positive integer; these axioms have been programmed into the machine in the form of certain instructions. The axioms are all well-known mathematical truths. The machine cannot prove any statement that is not a logical consequence of the axioms. Since the axioms are all true, and any logical consequence of true statements must be true, then the machine is incapable of proving a false sentence. I can tell you the axioms if you like, and then you can see for yourselves that the machine can prove only true sentences."

"Before you do that," said McCulloch, "I would like to ask another question. Suppose I am willing temporarily to take your word that every sentence provable by the machine is true. What about the converse? Is every true sentence of the form $x \in A_y$ provable by the machine? In other words, is the machine capable of proving *all* true sentences of the form $x \in A_y$, or only some?"

"A most important question," replied Fergusson, "but, alas, I don't know the answer! That is precisely the basic problem I have been unable to solve! I have been working on it on and off for months but have gotten nowhere. I know for sure that the machine can prove every statement $x \in A_y$ that is a logical consequence of the axioms, but I don't know whether I have programmed in enough axioms. The axioms

in question represent just about the sum total of what mathematicians know about the system of positive integers; still, there may not be enough to settle completely which numbers x belong to which nameable sets A_y. So far, every sentence $x \in A_y$ that I have examined and found to be true on purely mathematical grounds I have found to be a logical consequence of the axioms, and so the machine is capable of proving it. But just because I have not yet been able to find a true sentence that the machine cannot prove doesn't mean that there isn't one; it might be that I just haven't found it. Or, then again, it may be that the machine *can* prove all true sentences; but I have not yet been able to prove this fact. I just don't know!"

To make a long story short, at this point Fergusson told Craig and McCulloch all the axioms used by the machine, as well as the purely logical rules that enabled it to prove new sentences from old ones. Once Craig and McCulloch knew these details of the machine's operation, they could see immediately that it was indeed accurate—that it did prove only true sentences. But this still left unsolved the problem of whether the machine could prove all true sentences or only some. The three met together several times during the next few months and slowly but surely closed in on the problem, until they finally solved it.

I will not burden the reader with all the details, but will mention only those that are relevant to the solution of the problem. The turning point in the investigation came when the three men worked out three key properties of the machine; these properties sufficed to settle the question. It was, I believe, Craig and McCulloch who first brought the three properties to light, but it was Fergusson who applied the finishing touches. I will tell you what these properties are in a moment; but first, a little preliminary notation.

For any set A of positive integers, by its *complement* $\bar{A}$ is meant the set of all positive integers that are not in A. (For example, if A is the set of even numbers, then its complement $\bar{A}$ is the set of odd numbers; if A is the set of numbers divisible by 5, then its complement $\bar{A}$ is the set of numbers that are not divisible by 5.)

For any set A of positive integers, by A^* we shall mean the set of all positive integers x such that $x*x$ is a member of A. Thus, for any number x, to say that x lies in A^* is equivalent to saying that $x*x$ lies in A.

Now, here are the three key properties that Craig and McCulloch discovered about the machine:

Property 1: The set A_8 is the set of all numbers that the machine is capable of printing.

Property 2: For each positive integer n, $A_{3 \cdot n}$ is the complement of A_n. (By $3 \cdot n$ we mean 3 times n.)

Property 3: For every positive integer n, the set $A_{3 \cdot n+1}$ is the set A_n^* (the set of all numbers x such that $x*x$ belongs to A_n).

From Properties 1, 2, and 3, it can be rigorously deduced that Fergusson's machine is *not* able to prove all true sentences! The problem for the reader is to find a sentence that is true but not provable by the machine. That is, we are to find numbers n and m (either the same or different) such that n is in fact a member of the set A_m, yet the code number $n*m$ of the sentence $n \in A_m$ cannot possibly be printed by the machine.

PETER B. M. VRANAS
MWF 11-11:50, SPRING 2007

PRACTICE EXAM #1

Problem 1 (2 points). Please indicate on the answer sheet whether the following ten statements are true or false.

(a) An argument form is an ordered pair whose first member is a set of propositions (the premises) and whose second member is a proposition (the conclusion).
(b) The language of arithmetic has infinitely many constants.
(c) In official notation, every non-atomic term of the language of arithmetic ends with a right parenthesis.
(d) The empty language has infinitely many interpretations.
(e) The sentences '$\exists x \exists y(x \bullet y=1)$' and '$\exists x \exists y(x \bullet x=0)$' are equivalent over the standard interpretation of the language of arithmetic.
(f) In every interpretation of a language, every closed term denotes some element of the domain.
(g) '$\forall x \forall y(x+y=x+y)$' is a valid sentence of the language of arithmetic.
(h) The validity of a sentence is a semantic notion, but the demonstrability of a sentence is a syntactic notion.
(i) A proof procedure is complete exactly if every secure sequent is derivable.
(j) A set of sentences Γ is unsatisfiable exactly if $\Gamma \Rightarrow \varnothing$ is derivable.

Problem 2 (0.5 point). For each of the following five strings of symbols, write in the corresponding space of the answer sheet: 'A' if the string is not a term, 'B' if the string is an atomic closed term, 'C' if the string is an atomic open term, 'D' if the string is a non-atomic closed term, and 'E' if the string is a non-atomic open term (*in the language of arithmetic L*, using unofficial notation*).

(a) $2 \bullet z^2+y^2$
(b) $2+3$
(c) $(x+z')$
(d) $x<y$
(e) 0

Problem 3 (0.5 point). For each of the following five strings of symbols, write in the corresponding space of the answer sheet: 'A' if the string is not a formula, 'B' if the string is an atomic formula and a sentence, 'C' if the string is an atomic formula but not a sentence, 'D' if the string is a non-atomic formula and a sentence, and 'E' if the string is a non-atomic formula but not a sentence (*in the language of arithmetic L*, using unofficial notation*).

(a) $\forall x\ x=2 \vee x=3$
(b) $\forall y(y=2)$
(c) $\forall x(F(x) \vee \sim F(x))$
(d) $254 \neq 254$
(e) $2+(5 \bullet x)<(x'''+3) \bullet y$

Problem 4 (0.5 point). Translate into logical notation the following *definition*, using the notation provided: A pure imperative argument is *valid* exactly if every reason which supports the conjunction of the premises of the argument also supports the conclusion of the argument. (*Px*: *x* is a pure imperative argument; *Vx*: *x* is valid; *Rx*: *x* is a reason; *Sxy*: *x* supports *y*; *Cxy*: *x* is the conjunction of the premises of *y*; *Lxy*: *x* is the conclusion of *y*.)

Problem 5 (0.5 point). Provide a derivation (using (R0)-(R9)) of:
$\varnothing \Rightarrow \forall x(F(x) \vee \sim F(x))$.

PHILOSOPHY 511:
SYMBOLIC LOGIC

THE UNIVERSITY WISCONSIN MADISON

PETER B. M. VRANAS
MWF 11-11:50, SPRING 2007

EXAM #1

Problem 1 (2 points). Please indicate on the answer sheet whether the following ten statements are true or false.

(a) An argument is logically invalid exactly if it instantiates at least one invalid logical form.
(b) When the identity sign is *not* present, the empty language has infinitely many terms.
(c) When formulas are written in official notation, every formula ends with a right parenthesis.
(d) In every interpretation of the language of arithmetic, the identity sign denotes the relation of identity between natural numbers.
(e) In every interpretation of a language, every sentence of the language is either true or false.
(f) In every interpretation of the empty language, every element of the domain is the denotation of some closed term.
(g) '$\forall x \forall y(x\text{-}y=x\text{-}y)$' is a valid sentence of the language of arithmetic.
(h) Inconsistency is a syntactic notion, but unsatisfiability is a semantic notion.
(i) A proof procedure is sound exactly if every secure sequent is derivable.
(j) A sentence D is valid exactly if $\varnothing \Rightarrow \{D\}$ is derivable.

Problem 2 (0.5 point). For each of the following five strings of symbols, write in the corresponding space of the answer sheet: 'A' if the string is not a term, 'B' if the string is an atomic closed term, 'C' if the string is an atomic open term, 'D' if the string is a non-atomic closed term, and 'E' if the string is a non-atomic open term (*in the language of arithmetic L*, using unofficial notation*).

(a) $2\bullet z+3$
(b) 2-3
(c) $2+z$
(d) 327
(e) x

Problem 3 (0.5 point). For each of the following five strings of symbols, write in the corresponding space of the answer sheet: 'A' if the string is not a formula, 'B' if the string is an atomic formula and a sentence, 'C' if the string is an atomic formula but not a sentence, 'D' if the string is a non-atomic formula and a sentence, and 'E' if the string is a non-atomic formula but not a sentence (*in the language of arithmetic L*, using unofficial notation*).

(a) $\forall x \exists y(x=z)$
(b) $(\exists y)y<3$
(c) $3+x'$
(d) 254=254
(e) $(\exists y(2<y)\&\forall x(x<y))$

Problem 4 (0.5 point). Translate into logical notation the following *definition*, using the notation provided: A reason *weakly supports* a given prescription exactly if it strongly supports some prescription whose context is the same as the context of the given prescription. (Rx: x is a reason; Px: x is a prescription; Wxy: x weakly supports y; Sxy: x strongly supports y; Cxy: x is the context of y.)

Problem 5 (0.5 point). Provide a derivation (using (R0)-(R9)) of:
$\varnothing \Rightarrow \forall x \exists y(x=y)$.

PRACTICE EXAM #2

Problem 1 (3 points). Please indicate on the answer sheet whether the following fifteen statements are true or false.

1. When formulas are written in official notation, every formula begins with '(', '~', '∀', or '∃'.
2. The sentences '$\exists x \forall y(y<x+1)$' and '$\forall y \exists x(x<y+1)$' are equivalent over the standard interpretation of the language of arithmetic.
3. The following sequent is secure: $\{\exists x(0<x)\} \Rightarrow \{\exists x(x=0), \forall x{\sim}(x=0)\}$.
4. There is a three-line derivation (using (R0)-(R9)) of: $\varnothing \Rightarrow \exists x(x=c)$.
5. The proof procedure which consists of (R0)-(R4) (without (R5)-(R9)) is complete.
6. If a finite set of sentences is satisfiable, then it has a finite model.
7. If a set of sentences has no enumerable model, then some finite subset of the set has no denumerable model.
8. The set whose members are all and only the negations of the sentences of PA is a theory.
9. If a theory is complete, then it is decidable exactly if it is axiomatizable.
10. There are infinitely many distinct inconsistent theories in the language of arithmetic.
11. If a theory is incomplete, then it has two models which assign different truth values to some sentence in the language of the theory.
12. In George and Velleman's system of Gödel numbering, for every proof there is an expression such that the Gödel number of the proof is the same as the Gödel number of the expression.
13. It is a consequence of the fixed point lemma that there is a sentence Q in the language of PA such that: $\text{PA} \vdash (Q \leftrightarrow (\log(\ulcorner Q \urcorner)<0))$.
14. If a decidable extension of PA is ω-consistent, then it contains its Gödel sentence.
15. For any theory T, the set of Gödel numbers of theorems of T is not representable.

Problem 2 (0.8 point). For each of the following eight numbers, please write in the corresponding space of the answer sheet: 'A' if the number is not the Gödel number of any expression, 'B' if the number is the Gödel number of (an expression which is) an atomic open term, 'C' if the number is the Gödel number of an atomic closed term, 'D' if the number is the Gödel number of a non-atomic open term, 'E' if the number is the Gödel number of a non-atomic closed term, 'F' if the number is the Gödel number of an atomic formula and a sentence, 'G' if the number is the Gödel number of an atomic formula but not a sentence, 'H' if the number is the Gödel number of a non-atomic formula and a sentence, 'I' if the number is the Gödel number of a non-atomic formula but not a sentence, and 'J' if the number is the Gödel number of an expression which is of none of the above kinds (in the language of PA, using unofficial notation and George and Velleman's system of Gödel numbering).

1. $2^{16}\,3^{14}\,5^{17}$
2. $2^{3}\,3^{8}\,5^{11}\,7^{15}\,11^{11}\,13^{9}$
3. 48
4. $2^{6}\,3^{10}\,5^{8}\,7^{3}\,11^{12}+1$
5. $2^{3}\,3^{8}\,5^{11}\,7^{10}\,11^{11}\,13^{9}$
6. 36
7. 10^{9}
8. $7^{11}\,5^{13}\,3^{11}\,2^{12}$

Problem 3 (0.2 point). Translate into logical notation the following *definition*, using the notation provided: A set is *heterological* exactly if its members are all and only those sets that do not have all of their subsets as members. (Sx: x is a set; Hx: x is heterological; Mxy: x is a member of y; Zxy: x is a subset of y.)

PETER B. M. VRANAS
MWF 11-11:50, SPRING 2007

EXAM #2

Problem 1 (3 points). Please indicate on the answer sheet whether the following fifteen statements are true or false.

1. An occurrence of a variable in a formula is *bound* exactly if it is part of a subformula which is immediately preceded by a quantifier symbol.
2. The formulas '$\exists x(x<y+1)$' and '$\exists x(x=y+1)$' are equivalent over the standard interpretation of the language of arithmetic.
3. A demonstration of a sentence D is a deduction of D from $\varnothing$.
4. There is a three-line derivation (using (R0)-(R9)) of: $c=d \Rightarrow d=c$.
5. The proof procedure which consists of (R0)-(R6) (without (R7)-(R9)) is sound.
6. If an infinite set of sentences is satisfiable, then it has a denumerable model.
7. If an infinite set of sentences is unsatisfiable, then some infinite proper subset of the set is unsatisfiable.
8. The set whose members are all and only those sentences of the language of arithmetic which are false in the standard interpretation is a theory.
9. Every inconsistent theory is axiomatizable.
10. If a theory is inconsistent, then no sentence of its language is undecidable in the theory.
11. If a theory has two models which assign different truth values to some sentence in the language of the theory, then the theory is incomplete.
12. In every system of Gödel numbering, no two distinct expressions have the same Gödel number.
13. It is a consequence of the fixed point lemma that there is a formula $P(x_1)$ such that: $\mathrm{PA} \vdash (G_{\mathrm{PA}} \leftrightarrow \sim P(\ulcorner G_{\mathrm{PA}} \urcorner))$.
14. If an axiomatizable extension of PA is consistent, then it does not contain the negation of its Gödel sentence.
15. Every extension of PA is undecidable.

Problem 2 (0.8 point). For each of the following eight numbers, please write in the corresponding space of the answer sheet: 'A' if the number is not the Gödel number of any expression, 'B' if the number is the Gödel number of (an expression which is) an atomic open term, 'C' if the number is the Gödel number of an atomic closed term, 'D' if the number is the Gödel number of a non-atomic open term, 'E' if the number is the Gödel number of a non-atomic closed term, 'F' if the number is the Gödel number of an atomic formula and a sentence, 'G' if the number is the Gödel number of an atomic formula but not a sentence, 'H' if the number is the Gödel number of a non-atomic formula and a sentence, 'I' if the number is the Gödel number of a non-atomic formula but not a sentence, and 'J' if the number is the Gödel number of an expression which is of none of the above kinds (in the language of PA, using unofficial notation and George and Velleman's system of Gödel numbering).

1. $2^{11}\, 3^{10}\, 5^{11}$
2. $2^{12}\, 3^{11}$
3. 18
4. $2^{6}\, 3^{16}\, 5^{3}\, 7^{8}\, 11^{16}\, 13^{13}\, 15^{11}\, 17^{9}$
5. $5^{16}\, 3^{13}\, 2^{11}$
6. $3^{10}\, 5^{13}\, 7^{8}\, 24$
7. 30^{6}
8. 2

Problem 3 (0.2 point). Translate into logical notation the following *definition*, using the notation provided: Two people are *doubly compatriots* exactly if they are citizens of the same two nations. (Px: x is a person; Dxy: x and y are doubly compatriots; Cxy: x is a citizen of y; Nx: x is a nation.)

PHILOSOPHY 511:
SYMBOLIC LOGIC

PETER B. M. VRANAS
MWF 11-11:50, SPRING 2007

PRACTICE FINAL EXAM

Problem 1 (3 points). Please indicate on the answer sheet whether the following twenty statements are true or false.

1. An *instance* of formula $F(x)$ is a formula of the form $F(t)$ for any term t.
2. For the proof procedure consisting of (R0)-(R9), for any set of sentences Γ, and for any sentence D, $\Gamma \Rightarrow \{D\}$ is derivable exactly if $\Gamma \cup \{\sim D\}$ is inconsistent.
3. There is a nine-line derivation (using (R0)-(R9)) of: $\exists x(Fx \vee Gx) \Rightarrow \exists xFx \vee \exists xGx$.
4. For any formula $F(x)$ and for any constant c, the sentences '$\exists x(x{=}c \;\&\; F(x))$' and '$F(c)$' are logically equivalent.
5. A refutation of a set of sentences Γ is a deduction of $\{\varnothing\}$ from Γ.
6. If every infinite proper subset of an infinite set of sentences is satisfiable, then the whole set of sentences is satisfiable.
7. If a set of sentences has no enumerable model, then some finite subset of the set has no enumerable model.
8. The set whose members are all and only those sentences of the language of arithmetic whose Gödel numbers (in George and Velleman's system of Gödel numbering) is less than 10^{100} is a theory.
9. The set whose members are all and only those sentences of the language of arithmetic which are true in the standard interpretation is an axiomatizable theory (assuming that it is a superset of PA).
10. If a theory T is an axiomatizable extension of PA and n is the Gödel number of a theorem of T, then PA $\vdash \text{Theorem}_T(S^n0)$.
11. It is a consequence of the fixed point lemma that there is a sentence Q in the language of arithmetic such that: PA $\vdash (Q \leftrightarrow (\ulcorner Q \urcorner > 0))$.
12. If a theory is a consistent, axiomatizable, and ω-inconsistent extension of PA, then there is a sentence of the language of arithmetic such that neither it nor its negation is in the theory.
13. The formula $\text{Theorem}_{PA}(x_1)$ represents the set of Gödel numbers of theorems of PA.
14. A theory (in the language of arithmetic) is consistent exactly if it does not contain the sentence '$0{=}S0$'.
15. The axiomatizable theory whose axioms are those of PA together with the negation of the Gödel sentence of PA is undecidable if PA is consistent.
16. A function from natural numbers to natural numbers is Turing computable exactly if it is both abacus computable and recursive.
17. According to intuitionism, every sentence (in the language of arithmetic) that involves unbounded existential quantification is meaningful.
18. According to George and Velleman, Hilbert's project proceeds from a perspective that strips classical mathematics of all interpretation.
19. According to Franzén, PA_ω is subject to the incompleteness theorem.
20. According to Gaifman, the argument from Gödel's result to the no-computer thesis can be made without following McCall's route.

Problem 2 (0.8 point). For each of the following eight numbers, please write in the corresponding space of the answer sheet: 'A' if the number is not the Gödel number of any expression, 'B' if the number is the Gödel number of (an expression which is) an atomic open term, 'C' if the number is the Gödel number of an atomic closed term, 'D' if the number is the Gödel number of a non-atomic open term, 'E' if the number is the Gödel number of a non-atomic closed term, 'F' if the number is the Gödel number of an atomic formula and a sentence, 'G' if the number is the Gödel number of an atomic formula but not a sentence, 'H' if the number is the Gödel number of a non-atomic formula and a sentence, 'I' if the number is the Gödel number of a non-atomic formula but not a sentence, and 'J' if the number is the Gödel number of an expression which is of none of the above kinds (in the language of PA, using unofficial notation and George and Velleman's system of Gödel numbering).

1. 2^0
2. 258
3. $3^{15} \bullet 24$
4. $10^{16} \bullet 45$
5. $10^7 \bullet 13^{13} \bullet 21^{16} \bullet 187^{11}$
6. $2^3 \bullet 3^8 \bullet 5^{11} \bullet 7^{13} \bullet 11^{11} \bullet 13^9$
7. $2^6 \bullet 3^{16} \bullet 5^8 \bullet 7^{16} \bullet 11^{10} \bullet 13^{11}$
8. $2^7 \bullet 3^{16} \bullet 5^6 \bullet 7^{17} \bullet 11^3 \bullet 13^8 \bullet 17^{16} \bullet 19^{10} \bullet 21^{17} \bullet 23^9$

Problem 3 (0.2 point). Translate into logical notation the following sentence, using the notation provided: If every event has at least one cause which is also a cause of at least one different event, then no two distinct events have a common cause. (*Ex*: *x* is an event; *Cxy*: *x* is a cause of *y*.)

FINAL EXAM

Problem 1 (3 points). Please indicate on the answer sheet whether the following twenty statements are true or false.

1. An argument was defined to be *valid* exactly if it instantiates at least one valid argument form.
2. The formulas '$\forall x \exists y(x<z \ \& \ z<y)$' and '$\exists x \forall y(x<z \ \& \ z<y)$' are equivalent over the standard interpretation of the language of arithmetic.
3. There is a sixteen-line derivation (using (R0)-(R9)) of: $\exists x \exists y(Fx \vee Gy) \Rightarrow \sim(\sim\exists xFx \vee \sim\exists yGy)$.
4. For any set of sentences Γ, for any formula $F(x)$, and for any constant c, if $\Gamma \cup \{\sim\forall xF(x)\}$ is satisfiable, then $\Gamma \cup \{\sim F(c)\}$ is satisfiable.
5. For any sentences D, E, and F of a given language, $\{\sim E, \sim F\}$ implies D exactly if $\{\sim D\}$ secures $\{E, F\}$.
6. If every infinite proper subset of a set of sentences is satisfiable, then the whole set of sentences has an enumerable model.
7. If a set of sentences has no denumerable model, then there is a natural number n such that the set has no model of size greater than n.
8. There is a theory T (in the language of arithmetic) such that the set whose members are all and only those sentences of the language of arithmetic which are not in T is a theory.
9. The axiomatizable theory whose axioms are those of PA together with the consistency sentence of PA is undecidable regardless of whether PA is consistent.
10. The set whose members are all and only the Gödel numbers of formulas of the language of arithmetic which are *not* sentences is decidable.
11. It is a consequence of the fixed point lemma that there is a sentence Q of the language of arithmetic such that: PA $\vdash (Q \leftrightarrow \text{Proof}_{PA}(\ulcorner Q \urcorner, 0))$.
12. An extension T of PA is ω-inconsistent exactly if there is a formula $P(x_1)$ in the language of arithmetic such that: (a) $T \vdash \forall x_1 P(x_1)$ and (b) for every natural number m, $T \vdash \sim P(S^m 0)$.
13. According to the second part of Gödel's First Incompleteness Theorem, no ω-consistent extension of PA contains the negation of its Gödel sentence.
14. No consistent and axiomatizable extension of PA contains the sentence '$\sim(0=S0)$'.
15. There is no ω-consistent, decidable extension of PA.
16. According to Church's thesis, every recursively computable function is effectively computable.
17. According to finitism, every sentence (of the language of arithmetic) that involves unbounded quantification is meaningless.
18. According to George and Velleman, the belief that the Law of the Excluded Middle is correct is only peripheral to Hilbert's project.
19. According to Franzén, Lucas wrongly claims that Gödel's theorem states that in any consistent system which is strong enough to produce simple arithmetic there are formulas which cannot be proved in the system.
20. According to Gaifman, Gödel's result does not show that self-reflection cannot encompass the whole of our reasoning.

Problem 2 (0.8 point). For each of the following eight numbers, please write in the corresponding space of the answer sheet: 'A' if the number is not the Gödel number of any expression, 'B' if the number is the Gödel number of (an expression which is) an atomic open term, 'C' if the number is the Gödel number of an atomic closed term, 'D' if the number is the Gödel number of a non-atomic open term, 'E' if the number is the Gödel number of a non-atomic closed term, 'F' if the number is the Gödel number of an atomic formula and a sentence, 'G' if the number is the Gödel number of an atomic formula but not a sentence, 'H' if the number is the Gödel number of a non-atomic formula and a sentence, 'I' if the number is the Gödel number of a non-atomic formula but not a sentence, and 'J' if the number is the Gödel number of an expression which is of none of the above kinds (in the language of PA, using unofficial notation and George and Velleman's system of Gödel numbering).

1. 10^0	5. $42^{11} \bullet 7^{22} \bullet 50$
2. 1024	6. $30^{10} \bullet 320$
3. $6^{11} \bullet 2$	7. $2^8 \bullet 3^4 \bullet 5^{16} \bullet 7^{15} \bullet 9^4 \bullet 11^{17} \bullet 13^9 \bullet 17^{14} \bullet 19^{18} \bullet 23^9$
4. $30^{11} \bullet 9$	8. $2^7 \bullet 3^{10} \bullet 7^{17} \bullet 11^8 \bullet 13^{16} \bullet 15^6 \bullet 17^{10} \bullet 19^{12} \bullet 23^{17} \bullet 27^9$

Problem 3 (0.2 point). Translate into logical notation the following sentence, using the notation provided: Only police officers are allowed to arrest all and only those persons who have committed at least two crimes. (*Ox*: *x* is a police officer; *Px*: *x* is a person; *Rx*: *x* is a crime; *Lxy*: *x* is allowed to arrest *y*; *Cxy*: *x* has committed *y*.)

PHILOSOPHY 511:
SYMBOLIC LOGIC

PETER B. M. VRANAS
MWF 11:00-11:50, SPRING 2007

Name: Peter Vranas

ANSWER SHEET FOR PROBLEM SET #1

(Due Monday 29 January at 11:00 am in class)

PROBLEM 1

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
A	D	E	D	A	C	A	E

PROBLEM 2

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
D	A	C	B	D	C	D	E

PROBLEM 3

(a)	(b)	(c)	(d)
1x, 2x	3y	—	3z

PROBLEM 4

(a)	(b)
y is an uncle or the father of x	y is a niece or a daughter of x

ANSWER SHEET FOR QUIZ #1

(To be filled in at the beginning of class)

1	2	3	4
TRUE	FALSE	FALSE	TRUE

PETER B. M. VRANAS
MWF 11:00-11:50, SPRING 2007

Name: Peter Vranas

ANSWER SHEET FOR PROBLEM SET #2

(Due Monday 5 February at 11:00 am in class)

PROBLEM 1

Domain	Set of (human) persons
Denotation of P	Relation of mother to child
Denotation of Q	Relation of father to child
Denotation of R	Relation of parent to child
Denotation of a	Sarah
Denotation of b	Isaac
Denotation of c	Jacob

PROBLEM 2

	Interpretation 1	Interpretation 2
Domain	Set of (human) persons	Set of natural numbers
Denotation of P	Relation of uncle to nephew or niece	Being a multiple of
Denotation of Q	Relation of sibling to sibling	Being the square of
Denotation of a	Isaac	The number 2

PROBLEM 3

#	45	47	48	50	51	54	57	58
Truth val. on I_1	TRUE	FALSE	TRUE	FALSE	TRUE	FALSE	FALSE	TRUE
Truth val. on I_2	TRUE	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE
Truth val. on I_3	TRUE	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE

PROBLEM 4

(1) $\forall x\{Sx \rightarrow \forall y\{Pyx \leftrightarrow \exists z\{[Qz \,\&\, \forall w(Twz \rightarrow Mwx)] \,\&\, Lyz]\}\}\}$

(2) $\forall x\{Sx \rightarrow [Cx \leftrightarrow \forall y(Pyx \rightarrow Myx)]\}$

(3) $\forall x\{Kx \leftrightarrow [Sx \,\&\, \forall y(Myx \rightarrow Sy)]\}$

(4) $\forall x\{Kx \rightarrow \forall y\{Iyx \leftrightarrow \{Sy \,\&\, \forall z[Mzy \leftrightarrow \forall w(Mwx \rightarrow Mzw)]\}\}\}$

(5) $\forall x\{[Kx \,\&\, \forall y(Myx \rightarrow Cy)] \rightarrow \forall y(Iyx \rightarrow Cy)\}$

ANSWER SHEET FOR QUIZ #2

(To be filled in at the beginning of class)

1	2	3	4
FALSE	TRUE	FALSE	FALSE

PHILOSOPHY 511:
SYMBOLIC LOGIC

THE UNIVERSITY WISCONSIN MADISON

PETER B. M. VRANAS
MWF 11:00-11:50, SPRING 2007

Name: Peter Vranas

ANSWER SHEET FOR PROBLEM SET #3

(Due Monday 12 February at 11:00 am in class)

PROBLEM 1

(a) Take any interpretation that makes every sentence in Γ true. Since Γ implies every sentence in Δ, such an interpretation makes every sentence in Δ true. Since Δ implies E, such an interpretation makes E true. So Γ implies E.

(b) Take any interpretation that makes every sentence in Γ true. Since Γ implies every sentence in Δ, such an interpretation makes every sentence in $\Gamma \cup \Delta$ true. Since $\Gamma \cup \Delta$ implies E, such an interpretation makes E true. So Γ implies E.

PROBLEM 2

(a) $\sim C_1 \vee \cdots \vee \sim C_m$ is valid iff no interpretation makes it false; i.e., iff no interpretation makes all of $\sim C_1, \ldots, \sim C_m$ false; i.e., iff no interpretation makes all of $C_1, \ldots, C_m$ true; i.e., iff $\{C_1, \ldots, C_m\}$ is unsatisfiable.

(b) $\sim C_1 \vee \cdots \vee \sim C_m \vee D$ is valid iff no interpretation makes it false; i.e., iff no interpretation makes all of $\sim C_1, \ldots, \sim C_m, D$ false; i.e., iff no interpretation makes all of $C_1, \ldots, C_m$ true and D false; i.e., iff D is a consequence of $\{C_1, \ldots, C_m\}$.

(c) $\{C_1, \ldots, C_m, \sim D\}$ is unsatisfiable iff $\sim C_1 \vee \cdots \vee \sim C_m \vee \sim\sim D$ is valid [from (a) above]; i.e., iff $\sim C_1 \vee \cdots \vee \sim C_m \vee D$ is valid; i.e., iff D is a consequence of $\{C_1, \ldots, C_m\}$ [from (b) above].

(d) D is valid iff no interpretation makes D false; i.e., iff no interpretation makes $\sim D$ true; i.e., iff $\sim D$ is unsatisfiable.

PROBLEM 3

$\forall x (F(x) \leftrightarrow G(x))$ is valid iff every interpretation makes it true; i.e., iff for every interpretation $\mathcal{M}$, and for every m in the domain of $\mathcal{M}$, (in $\mathcal{M}$) m satisfies $F(x) \leftrightarrow G(x)$; i.e., iff for every interpretation $\mathcal{M}$, and for every m in the domain of $\mathcal{M}$, $\mathcal{M}^c_m$ makes $F(c) \leftrightarrow G(c)$ true; i.e., iff for every interpretation $\mathcal{M}$, F(x) and G(x) are equivalent over $\mathcal{M}$; i.e., iff F(x) and G(x) are logically equivalent.

PROBLEM 4

$\Gamma \cup \sim\Delta$ is unsatisfiable iff no interpretation makes all of its members true; i.e., iff no interpretation makes every member of Γ true and every member of Δ false; i.e., iff every interpretation that makes every member of Γ true does not make every member of Δ false; i.e., iff every interpretation that makes every member of Γ true makes at least one member of Δ true; i.e., iff Γ secures Δ.

ANSWER SHEET FOR QUIZ #3

(To be filled in at the beginning of class)

1	2	3	4
FALSE	FALSE	FALSE	FALSE

PHILOSOPHY 511:
SYMBOLIC LOGIC

THE UNIVERSITY WISCONSIN MADISON

PETER B. M. VRANAS
MWF 11:00-11:50, SPRING 2007

Name: Peter Vranas

ANSWER SHEET FOR PROBLEM SET #4

(Due Monday 19 February at 11:00 am in class)

PROBLEM 1

$$\frac{c \neq c \Rightarrow}{\exists x (x \neq c) \Rightarrow}$$

PROBLEM 2

(1)	$R(c,d) \Rightarrow R(c,d)$	(R0)
(2)	$\Rightarrow R(c,d), \sim R(c,d)$	(R2a), (1)
(3)	$\Rightarrow R(c,d), \exists x \sim R(x,d)$	(R5), (2)
(4)	$\Rightarrow \exists y R(c,y), \exists x \sim R(x,d)$	(R5), (3)
(5)	$\sim \exists y R(c,y) \Rightarrow \exists x \sim R(x,d)$	(R2b), (4)
(6)	$\exists x \sim \exists y R(x,y) \Rightarrow \exists x \sim R(x,d)$	(R6), (5)
(7)	$\exists x \sim \exists y R(x,y), \sim \exists x \sim R(x,d) \Rightarrow$	(R2b), (6)
(8)	$\exists x \sim \exists y R(x,y), \exists y \sim \exists x \sim R(x,y) \Rightarrow$	(R6), (7)
(9)	$\exists y \sim \exists x \sim R(x,y) \Rightarrow \sim \exists x \sim \exists y R(x,y)$	(R2a), (8)
(10)	$\exists y \forall x R(x,y) \Rightarrow \forall x \exists y R(x,y)$	(9)

PROBLEM 3

(1)	$F(a), G(b) \Rightarrow F(a) \,\&\, G(b)$	Class discussion
(2)	$F(a), G(b) \Rightarrow \exists y (F(a) \,\&\, G(y))$	(R5), (1)
(3)	$F(a), G(b) \Rightarrow \exists x \exists y (F(x) \,\&\, G(y))$	(R5), (2)
(4)	$\exists x F(x), G(b) \Rightarrow \exists x \exists y (F(x) \,\&\, G(y))$	(R6), (3)
(5)	$\exists x F(x), \exists y G(y) \Rightarrow \exists x \exists y (F(x) \,\&\, G(y))$	(R6), (4)
(6)	$\exists y G(y) \Rightarrow \exists x \exists y (F(x) \,\&\, G(y)), \sim \exists x F(x)$	(R2a), (5)
(7)	$\Rightarrow \exists x \exists y (F(x) \,\&\, G(y)), \sim \exists x F(x), \sim \exists y G(y)$	(R2a), (6)
(8)	$\Rightarrow \exists x \exists y (F(x) \,\&\, G(y)), \sim \exists x F(x) \vee \sim \exists y G(y)$	(R3), (7)
(9)	$\sim(\sim \exists x F(x) \vee \sim \exists y G(y)) \Rightarrow \exists x \exists y (F(x) \,\&\, G(y))$	(R2b), (8)
(10)	$\exists x F(x) \,\&\, \exists y G(y) \Rightarrow \exists x \exists y (F(x) \,\&\, G(y))$	(9)

PROBLEM 4

(a) The proof procedure consisting just of the rule (R0) is sound (since only sequents of the form $\{A\} \Rightarrow \{A\}$ are derivable, and all such sequents are secure) but is not complete (since only sequents of the form $\{A\} \Rightarrow \{A\}$ are derivable, so some secure sequents — e.g., $\{A\} \Rightarrow \{A \vee B\}$ — are not derivable)

(b) The proof procedure consisting just of the rule $\frac{}{\Gamma \Rightarrow \Delta}$ is complete (since every sequent is derivable, so every secure sequent is derivable) but is not sound (since every non-secure sequent — e.g., $\{A \vee B\} \Rightarrow \{A\}$ — is also derivable).

ANSWER SHEET FOR QUIZ #4

(To be filled in at the beginning of class)

1	2	3	4
Every secure sequent is derivable	$\Gamma \Rightarrow \emptyset$	TRUE	FALSE

Name: Peter Vranas

ANSWER SHEET FOR PROBLEM SET #5

(Due Monday 5 March at 11:00 am in class)

PROBLEM 1

Since $\Gamma_1 \cup \Gamma_2$ is unsatisfiable, by the compactness theorem some finite subset Γ_0 of $\Gamma_1 \cup \Gamma_2$ is unsatisfiable. Let $\Gamma_1' = \Gamma_0 \cap \Gamma_1$ and $\Gamma_2' = \Gamma_0 \cap \Gamma_2$. Then $\Gamma_1' \cup \Gamma_2' = \Gamma_0$, so both Γ_1' and Γ_2' are finite. Moreover, at least one of Γ_1', Γ_2' is nonempty (otherwise their union, namely Γ_0, would be empty and thus satisfiable); without loss of generality, suppose $\Gamma_1' \neq \emptyset$. Since Γ_1' is finite, let S be the conjunction of its members. Since $\Gamma_1' \subseteq \Gamma_1$, $\Gamma_1 \vDash S$. Since $\Gamma_1' \cup \Gamma_2'$ is unsatisfiable, every interpretation that makes Γ_2' true makes at least one sentence in Γ_1' false, and thus makes $\sim S$ true, so $\Gamma_2' \vDash \sim S$. Since $\Gamma_2' \subseteq \Gamma_2$, $\Gamma_2 \vDash \sim S$. □

PROBLEM 2

Counterexample: γ_1 = '$P(a)$', γ_2 = '$P(b)$', γ_3 = '$\sim P(a) \vee \sim P(b)$'. Then $\{\gamma_1, \gamma_2\}$ is satisfiable (let a denote the number 2, b denote the number 4, and P denote the property of being even), $\{\gamma_2, \gamma_3\}$ is satisfiable (take an interpretation as before, but with a denoting the number 3), and $\{\gamma_1, \gamma_3\}$ is satisfiable (take an interpretation as the first one above, but with b denoting the number 3), but $\{\gamma_1, \gamma_2, \gamma_3\}$ is unsatisfiable (because every interpretation that makes γ_3 true makes at least one of $P(a)$, $P(b)$ (i.e., γ_1, γ_2) false).

ANSWER SHEET FOR QUIZ #5

(To be filled in at the beginning of class)

1	2	3	4
TRUE	FALSE	FALSE	FALSE

PHILOSOPHY 511:
SYMBOLIC LOGIC

THE UNIVERSITY of WISCONSIN MADISON

PETER B. M. VRANAS
MWF 11:00-11:50, SPRING 2007

Name: Peter Vranas

ANSWER SHEET FOR PROBLEM SET #6

(Due Monday 12 March at 11:00 am in class)

PROBLEM 1

(a) Given the assumptions of the problem, 'I am not an established knight' is equivalent to 'I am a knave or a non-established knight'. So the inhabitant can neither be a knave (since then his statement would be true, whereas knaves always lie) nor be an established knight (since then his statement would be false, whereas knights never lie), and must thus be a non-established knight.
(b) The inhabitant can neither be a knight (since then his statement would be false, whereas knights never lie), nor be an established knave (since then his statement would be true, whereas knaves always lie), and must thus be a non-established knave (and his statement is false).
(c) The inhabitant cannot be a non-established knight (since then his statement would be false, whereas knights never lie), but may be either an established knight (and then his statement is true) or a (non-established or established) knave (and then his statement is false). So we cannot tell whether he is a knight or a knave and whether or not he is established.

PROBLEM 2

(a) Sentence: NPA-NPA. This says that the associate of NPA is not printable. But the associate of NPA is NPA-NPA. So the sentence says of itself that it is not printable.
(b) Proof that the sentence is true: The sentence cannot be false, since then it would be printable, and thus it would be true, given that every printable sentence is true. So the sentence is true.
(c) Proof that the machine cannot print the sentence: From (b), the sentence is true. Moreover, the sentence says that it is not printable. So it is not printable.

ANSWER SHEET FOR QUIZ #6

(To be filled in at the beginning of class)

1	2	3	4
FALSE	FALSE	TRUE	FALSE

Name: Peter Vranas

ANSWER SHEET FOR PROBLEM SET #7

(Due Monday 19 March at 11:00 am in class)

PROBLEM 1

Sentence: $\forall\ x_1\ (\ (\ x_1\ \cdot\ 0\ )\ =\ 0\ )$

Sequence of Gödel numbers of the symbols in the sentence: $\langle 5, 15, 7, 7, 15, 14, 10, 8, 9, 10, 8 \rangle$

Code number of the sequence (= Gödel number of the sentence): $2^6 \cdot 3^{16} \cdot 5^8 \cdot 7^8 \cdot 11^{16} \cdot 13^{15} \cdot 17^{11} \cdot 19^9 \cdot 23^{10} \cdot 29^{11} \cdot 31^9$

PROBLEM 2

Prime factorization of the Gödel number: $2^6 \cdot 3^{16} \cdot 5^3 \cdot 7^8 \cdot 11^{12} \cdot 13^{16} \cdot 17^{10} \cdot 19^{11} \cdot 23^9$

Corresponding sequence: $\langle 5, 15, 2, 7, 11, 15, 9, 10, 8 \rangle$

Corresponding expression: $\forall\ x_1\ \sim\ (\ S\ x_1\ =\ 0\ )$

PROBLEM 3

$P(x_1, x_2)$: $(0 < x_1)\ \&\ \exists x_3 (P_i(x_3, x_1)\ \&\ P_{ii}(x_3, S0, S0)\ \&$
$\forall x_4 ((((S0 < x_4) \vee (S0 = x_4))\ \&\ (x_4 < x_1)) \rightarrow \exists x_5 (P_{ii}(x_3, x_4, x_5)\ \&$
$\exists x_6 (P_{ii}(x_3, Sx_4, x_6)\ \&\ (x_6 = (x_5 \cdot Sx_4))\ \&\ P_{ii}(x_3, x_1, x_2))))).$

ANSWER SHEET FOR QUIZ #7

(To be filled in at the beginning of class)

1	2	3	4
FALSE	TRUE	FALSE	TRUE

PHILOSOPHY 511:
SYMBOLIC LOGIC

PETER B. M. VRANAS
MWF 11:00-11:50, SPRING 2007

Name: Peter Vranas

ANSWER SHEET FOR PROBLEM SET #8

(Due Monday 26 March at 11:00 am in class)

PROBLEM 1

(a) T is consistent because there is a sentence, namely G_{PA}, that T does not contain. Indeed, suppose for reductio that $T \vdash G_{PA}$. Then $PA \cup \{\sim G_{PA}\} \vdash G_{PA}$, so (e.g., by rule (R9a) of the sequent calculus) $PA \vdash G_{PA}$, contradicting Gödel's First Incompleteness Theorem (since PA is a decidable extension of PA and is assumed to be consistent).

(b) Since by definition $T \vdash \neg G_{PA}$ and from the fixed point lemma PA, and thus its extension T, contains '$G_{PA} \leftrightarrow \neg Theorem_{PA}(\ulcorner G_{PA} \urcorner)$', from the closure of T under logical consequence we get $T \vdash Theorem_{PA}(\ulcorner G_{PA} \urcorner)$; i.e., (1) $T \vdash \exists x_2 Proof_{PA}(\ulcorner G_{PA} \urcorner, x_2)$. Since every proof in PA is also a proof in T, (1) gives: (2) $T \vdash \exists x_2 Proof_{T}(\ulcorner G_{PA} \urcorner, x_2)$. But we know from (a) above that G_{PA} is not a theorem of T, so no natural number m can be the Gödel number of a proof of G_{PA} in T: (3) for every number m, $T \vdash \neg Proof_{T}(\ulcorner G_{PA} \urcorner, S^m 0)$. The combination of (2) and (3) amounts to the ω-inconsistency of T. (This step uses Theorem 7.12.)

PROBLEM 2

(a) Sentence: $73 \in A_{73}$

(b) Proof that the sentence is true: Suppose, for reductio, that the sentence is false; i.e., that $73 \notin A_{73}$. From Property 3, $73 \in A_{73} \leftrightarrow 73*73 \in A_{24}$ (since $73 = 3 \cdot 24 + 1$), so from $73 \notin A_{73}$ we get $73*73 \notin A_{24}$. So $73*73 \in \overline{A_{24}} = A_8$ (from Property 2, since $24 = 3 \cdot 8$), so (from Property 1) $73*73$ is printable, so $73 \in A_{73}$ after all, and the reductio is complete.

(c) Proof that the machine cannot prove the sentence: We saw above that $73 \in A_{73}$ and that $73 \in A_{73} \leftrightarrow 73*73 \in A_{24}$. It follows that $73*73 \in A_{24} = \overline{A_8}$, so $73*73 \notin A_8$; thus $73*73$ is unprintable, and '$73 \in A_{73}$' is unprovable.

ANSWER SHEET FOR QUIZ #8

(To be filled in at the beginning of class)

1	2	3	4
TRUE	FALSE	FALSE	TRUE

PHILOSOPHY 511:
SYMBOLIC LOGIC

THE UNIVERSITY of WISCONSIN MADISON

PETER B. M. VRANAS
MWF 11:00-11:50, SPRING 2007

Name: Peter Vranas

ANSWER SHEET FOR PROBLEM SET #9

(Due Monday 9 April at 11:00 am in class)

PROBLEM 1

Statement: 'You cannot prove that I am a knight'. The native cannot be a knave, because then the statement would be false, so the logician could prove that the native is a knight, and thus could prove something false, contrary to the assumption that the logician is accurate. So the native must be a knight; thus the statement is true and the logician cannot prove that the native is a knight.

PROBLEM 2

In the above solution to Problem 1, the proof that the native is a knight uses the assumption that the logician is accurate. But although it is by assumption true that the logician is accurate, it is not assumed that the logician knows that she is accurate, so she may not use this assumption to go through the above proof herself. Indeed, the logician cannot prove that she is accurate: if she could, then she could go through the above proof and prove that the native is a knight, but this would falsify the native's statement, so he would be a knave and the logician would be inaccurate because she would have proved something false. The inability of an accurate logician to prove her own accuracy is analogous to the inability of a consistent and decidable extension of PA to prove its own consistency (Gödel's Second Incompleteness Theorem).

ANSWER SHEET FOR QUIZ #9

(To be filled in at the beginning of class)

1	2	3	4
FALSE	FALSE	TRUE	TRUE

PHILOSOPHY 511:
SYMBOLIC LOGIC

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ANSWER KEY FOR QUIZZES #10-18

Quiz	Statement 1	Statement 2	Statement 3	Statement 4
10	False	False	False	True
11	False	True	False	True
12	False	True	False	True
13	False	True	True	True
14	False	True	True	False
15	True	True	False	False
16	True	False	True	True
17	True	True	False	False
18	True	False	True	True

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ANSWER SHEET FOR PRACTICE EXAM #1

PROBLEM 1

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)
FALSE	FALSE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	FALSE

PROBLEM 2

(a)	(b)	(c)	(d)	(e)
A	D	E	A	B

PROBLEM 3

(a)	(b)	(c)	(d)	(e)
E	D	A	D	C

PROBLEM 4

$\forall x\{Px \to [Vx \leftrightarrow \forall y[(Ry \& \forall z(Czx \to Syz)) \to \forall w(Lwx \to Syw)]]\}$

PROBLEM 5

(1)	$F(c) \Rightarrow F(c)$	(R0)
(2)	$\Rightarrow F(c), \sim F(c)$	(R2a), (1)
(3)	$\Rightarrow F(c) \vee \sim F(c)$	(R3), (2)
(4)	$\sim (F(c) \vee \sim F(c)) \Rightarrow$	(R2b), (3)
(5)	$\exists x \sim (F(x) \vee \sim F(x)) \Rightarrow$	(R6), (4)
(6)	$\Rightarrow \sim \exists x \sim (F(x) \vee \sim F(x))$	(R2a), (5)
(7)	$\Rightarrow \forall x (F(x) \vee \sim F(x))$	(6)

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ANSWER SHEET FOR EXAM #1

PROBLEM 1

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)
FALSE	TRUE	TRUE	FALSE	TRUE	FALSE	FALSE	TRUE	FALSE	FALSE

PROBLEM 2

(a)	(b)	(c)	(d)	(e)
E	A	E	D	C

PROBLEM 3

(a)	(b)	(c)	(d)	(e)
E	A	A	B	E

PROBLEM 4

$$\forall x \forall y \{(Rx \& Py) \rightarrow [Wxy \leftrightarrow \exists z(Pz \& Sxz \& \forall w \forall v((Cwy \& Cvz) \rightarrow w=v))]\}$$

PROBLEM 5

(1)	$c=c \Rightarrow c=c$	(R0)
(2)	$\Rightarrow c=c$	(R7), (1)
(3)	$\Rightarrow \exists y(c=y)$	(R5), (2)
(4)	$\sim\exists y(c=y) \Rightarrow$	(R2b), (3)
(5)	$\exists x \sim\exists y(x=y) \Rightarrow$	(R6), (4)
(6)	$\Rightarrow \sim\exists x \sim\exists y(x=y)$	(R2a), (5)
(7)	$\Rightarrow \forall x \exists y(x=y)$	(6)

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ANSWER SHEET FOR PRACTICE EXAM #2

PROBLEM 1

1	2	3	4	5
FALSE	FALSE	TRUE	TRUE	FALSE
6	7	8	9	10
FALSE	TRUE	FALSE	TRUE	FALSE
11	12	13	14	15
TRUE	TRUE	FALSE	TRUE	FALSE

PROBLEM 2

1	2	3	4
D	J	J	A
5	6	7	8
H	J	A	F

PROBLEM 3

$\forall x(Sx \rightarrow (Hx \leftrightarrow \forall y(Myx \leftrightarrow (Sy \,\&\, \sim\forall z(Zzy \rightarrow Mzy)))))$

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ANSWER SHEET FOR EXAM #2

PROBLEM 1

1	2	3	4	5
FALSE	TRUE	TRUE	TRUE	TRUE
6	7	8	9	10
FALSE	TRUE	FALSE	TRUE	TRUE
11	12	13	14	15
TRUE	TRUE	FALSE	FALSE	FALSE

PROBLEM 2

1	2	3	4
F	E	J	J
5	6	7	8
G	J	J	J

PROBLEM 3

$\forall x \forall y((Px \& Py) \rightarrow (Dxy \leftrightarrow (\exists v \exists w(Nv \& Nw \& v \neq w \& Cxv \& Cxw \& Cyv \& Cyw)))$

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ANSWER SHEET FOR PRACTICE FINAL EXAM

PROBLEM 1

1	2	3	4	5
FALSE	TRUE	TRUE	TRUE	FALSE
6	7	8	9	10
TRUE	TRUE	FALSE	FALSE	TRUE
11	12	13	14	15
FALSE	TRUE	FALSE	FALSE	TRUE
16	17	18	19	20
TRUE	TRUE	TRUE	TRUE	TRUE

PROBLEM 2

1	2	3	4
A	A	J	J
5	6	7	8
H	H	J	J

PROBLEM 3

$\forall x(Ex \rightarrow \exists y((yx \& \exists z(z \neq x \& (yz))) \rightarrow \sim \exists x \exists y(Ex \& Ey \& x \neq y \& \exists z((zx \& (zy))$

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ANSWER SHEET FOR FINAL EXAM

PROBLEM 1

1	2	3	4	5
FALSE	TRUE	FALSE	FALSE	TRUE
6	**7**	**8**	**9**	**10**
FALSE	TRUE	FALSE	FALSE	TRUE
11	**12**	**13**	**14**	**15**
TRUE	FALSE	FALSE	FALSE	TRUE
16	**17**	**18**	**19**	**20**
FALSE	FALSE	FALSE	FALSE	FALSE

PROBLEM 2

1	2	3	4
A	J	E	F
5	**6**	**7**	**8**
J	G	J	J

PROBLEM 3

$\forall x(\forall y(Lxy \leftrightarrow (Py \,\&\, \exists z \exists w(Rz \,\&\, Rw \,\&\, z \neq w \,\&\, Cyz \,\&\, Cyw))) \rightarrow Ox)$