PHILOSOPHY 511:
SYMBOLIC LOGIC

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MWF 11-11:50, FALL 2021

PROBLEM SETS

PROBLEM SET #1

Due: Monday 13 September at 11:00am in class

Problem 1 (1 point). For the following eight strings of symbols, write on the answer sheet: ‘A’ if the string is not a term, ‘B’ if the string is an atomic closed term, ‘C’ if the string is an atomic open term, ‘D’ if the string is a non-atomic closed term, and ‘E’ if the string is a non-atomic open term (*in the language of arithmetic L*; unofficial notation acceptable*).

(a) $z+z$ (c) $0{\bullet}x$ (e) $0=y$ (g) 0-1
(b) $5+x<3{\bullet}(4+2)$ (d) $z \vee y$ (f) $+(y, 0)$ (h) $0\text{-}x$

Problem 2 (1 point). For the following eight strings of symbols, write on the answer sheet: ‘A’ if the string is not a formula, ‘B’ if the string is an atomic formula and a sentence, ‘C’ if the string is an atomic formula but not a sentence, ‘D’ if the string is a non-atomic formula and a sentence, and ‘E’ if the string is a non-atomic formula but not a sentence (*in the language of arithmetic L*; unofficial notation acceptable*).

(a) $\forall x\forall y(x>y)$ (d) $\forall x\forall z(x \ \& \ (\exists y(z<x) \vee x=z))$ (g) $y+x<x+y$
(b) $\forall x\exists y\forall y(x=y \ \& \ y<z)$ (e) $2+3{\bullet}(2{\bullet}x+3{\bullet}x)=x+(3{\bullet}x+z)$ (h) $55<4+z$
(c) $\forall x\exists y(x+x=y)$ (f) $\exists x(2<2)$

Problem 3 (1 point). For the following four formulas (in the language $L^* \cup \{P, Q, R\}$, where P and Q are two-place predicates and R is a one-place predicate; *official notation is used*), write on the answer sheet all free occurrences of variables in the following format: ^2x if the second occurrence of ‘x’ is free in the formula, and so on.

(a) $\forall z(P(x, z) \ \& \ \forall zQ(z, x))$ (c) $\forall z{\sim}\exists y(P(x, y) \ \& \ \forall xR(x)) \vee {\sim}(R(z) \ \& \ \forall xR(y))$
(b) $\forall z(\exists xP(x, z) \ \& \ R(x))$ (d) $(Q(y, z) \ \& \ \forall y(Q(z, 0) \vee {\sim}\exists z(P(z, z) \ \& \ \forall y\exists yR(z))))$

Problem 4 (1 point). Do problem 9.1 on p. 112 of BBJ.

PROBLEM SET #2

Due: Monday 20 September at 11:00am in class

Problem 1 (1 point). Do Problem 9.2 on p. 112 of BBJ *modified* as follows: give an interpretation that makes (9) false, (10) true, (11) false, and (12) true.

Problem 2 (1 point). Do exercise D on p. 239 of Bessie & Glennan (see next pages), *only* #66.

Problem 3 (1 point). Do exercise C on pp. 237-239 of Bessie & Glennan (see next pages), *only* #36, 41, 44, 46, 49, 53, 55, 56.

Problem 4 (1 point). Translate: (1) A *theory* is a set of sentences that contains every sentence provable from the set. (2) A set of sentences is *decidable* iff there is an effective procedure that, for every sentence, correctly decides whether the sentence is or not contained in the set. (3) A theory is *axiomatizable* iff it has a decidable subset such that every sentence contained in the theory is provable from that subset. (4) A theory is *complete* iff, for every sentence, either the sentence or its negation is contained in the theory. (5) If a theory is complete, then it is decidable iff it is axiomatizable. (6) A theory is *consistent* iff there is no sentence such that both it and its negation are provable from the theory. (*Tx*: *x* is a theory; *Sx*: *x* is a set of sentences; *Nx*: *x* is a sentence; *Dx*: *x* is decidable; *Ax*: *x* is axiomatizable; *Fx*: *x* is an effective procedure; *Lx*: *x* is complete; *Cx*: *x* is consistent; *Rxy*: *x* is provable from *y*; *Mxy*: *x* contains *y*; *Bxy*: *x* is a subset of *y*; *Zxyz*: *x* decides that *y* is contained in *z*; *Gxy*: *x* is the negation of *y*.)

FROM: BESSIE & GLENNAN, ELEMENTS OF DEDUCTIVE INFERENCE

Exercises for Section 5.5

(A) Below are listed three interpretations, followed by several statements involving monadic predicates. Evaluate the truth values of each statement, using $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$.

$\mathcal{I}_1$	$\mathcal{D}$:	animals
	Ax:	x is a mammal
	Bx:	x is a ferret
	Cx:	x is female
	a:	Mickey Mouse
	b:	Minnie Mouse
	c:	Donald Duck

$\mathcal{I}_2$	$\mathcal{D}$:	natural numbers {1, 2, 3, . . .}
	Ax:	x is even
	Bx:	x is odd
	Cx:	x is prime
	a:	2
	b:	3
	c:	4

$\mathcal{I}_3$	$\mathcal{D}$:	{Burt, Ernie, Elmo, Cookie Monster, Big Bird}
	Ax:	{Burt, Ernie}
	Bx:	{Burt, Elmo, Cookie Monster}
	Cx:	$\varnothing$
	a:	Big Bird
	b:	Elmo
	c:	Cookie Monster

1. Ca
2. $\sim(Aa \rightarrow Bb)$
3. ♦ $\sim(Ba \vee Ca)$
4. $\sim Ba \vee Ca$
5. $\exists xAx$
6. $\sim\forall x\sim Cx$
7. ♦ $\exists xAx \;\&\; \exists x\sim Bx$
8. $\forall x(Bx \leftrightarrow Ax)$
9. $\exists x(Ax \;\&\; Cx)$
10. $\exists x(Cx \vee \sim Cx)$
11. $\forall x(Bx \rightarrow Cx)$
12. $\exists x(Bx \rightarrow Cx)$
13. ♦ $\forall xBx \rightarrow Ca$
14. $(Ba \;\&\; Bb) \rightarrow \forall zBz$
15. $\forall x[(Ax \;\&\; Bx) \rightarrow \sim Cx]$

(B) For each of the following *L* statements involving monadic predicates, provide two interpretations: one under which the given statement is true and one under which the given statement is false.

Example: Given: $\forall x(Ax \rightarrow Bx)$

1. $\mathcal{D}$:	people		2. $\mathcal{D}$:	people
Ax:	x is a person		Ax:	x is a man
Bx:	x has a mother		Bx:	x is a father

Under interpretation 1 the statement says 'All people have mothers', which is true. Under interpretation 2 the statement says 'All men are fathers', which is false.

16. Ar
17. $\forall xAx \rightarrow \exists yBy$
18. $Br \;\&\; \forall xAx$
19. ♦ $Ar \;\&\; Br$
20. $\exists xAx \rightarrow \forall yBy$
21. $Mg \rightarrow \forall x(Bx \leftrightarrow Mx)$
22. $Ma \vee Fb$
23. $\exists xAx \rightarrow \forall xAx$
24. $(\exists xAx \;\&\; \exists xBx) \rightarrow \forall zKz$
25. ♦ $\exists x(Fx \vee Mx)$
26. $\exists zFz \vee \exists yGy$
27. $\forall x(Ax \rightarrow Bx) \rightarrow \exists x\sim Cx$
28. $\exists x(Fx \;\&\; Mx)$
29. $\sim\exists x(Ax \vee Bx)$
30. $\forall x((Ax \;\&\; Bx) \leftrightarrow Cx)$
31. ♦ $\exists xAx \rightarrow \forall xBx$
32. $\sim\forall x(Ax \rightarrow Bx)$
33. $\forall x[Fx \rightarrow (Gx \vee Hx)]$
34. $\forall y[(Ay \;\&\; By) \;\&\; Cy]$
35. $\forall xWx \rightarrow \forall z(Vz \vee \sim Pz)$

PROBLEM 3 OF PROBLEM SET #2

(C) Below are listed three interpretations, followed by several statements involving polyadic predicates. Evaluate the truth values of each statement, using $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$.

$\mathcal{I}_1$	$\mathcal{D}$:	natural numbers {1, 2, 3, . . .}
	Gxy:	x is greater than y
	Lxy:	x is less than y
	Sxyz:	the sum of x and y is z
	Pxyz:	the product of x and y is z
	Ex:	x is even
	Ox:	x is odd
	a:	1
	b:	2
	c:	3

FROM: BESSIE & GLENNAN, ELEMENTS OF DEDUCTIVE INFERENCE

$\mathcal{I}_2$

𝒟:	{Susan, Emily, Lucinda, Samantha, Kirk, Lily, Holden, James, David, Molly, Ben, Camille, Cal, Bob, Kim, Jack, Margo, Tom, Adam}
Gxy:	{<Lucinda, Lily>, <Kim, Tom>, <Susan, Emily>, <Margo, Adam>}
Lxy:	{<Tom, Adam>, <Bob, Tom>}
Sxyz:	{<Tom, Margo, Adam>, <Margo, Tom, Adam>, <Kim, Bob, Tom>, <Bob, Kim, Tom>}
Pxyz:	{<Lucinda, Samantha, Lily>, <Samantha, Lucinda, Lily>, <Lily, Lucinda, Samantha>, <Samantha, Lily, Lucinda>}
Ex:	{Holden, James, David, Ben, Cal, Bob, Tom, Jack}
Ox:	{Susan, Emily, Lucinda, Samantha, Lily, Molly, Camille, Kim, Margo}
a:	Lucinda
b:	Samantha
c:	Lily

$\mathcal{I}_3$

𝒟:	states of the USA
Gxy:	x is larger in area than y
Lxy:	x is more populous than y
Sxyz:	x is is between y and z (ie., east of y and west of z)
Pxyz:	x is bordered by y and z
Ex:	x is a coastal state
Ox:	x is one of the contiguous 48 states
a:	Alaska
b:	Maryland
c:	Illinois

36. Sabc
37. Pabb
38. Gca & Lac
♦39. Lab & ~Gac
40. ∃x(Ex & Gxb)
41. ∃x(Ox & Gxc)
42. ~∃xGxx
43. ∃xSabx
44. ∃x(Sccx & Ox)
♦45. ∀x∀y(Gxy → ~Gyx)
46. ∀x∀y(Gxy → Lyx)
47. ∀x∀y(~Gxy → Lxy)
48. ∀x∀y∃zSxyz
49. ∀x∃yGxy
50. ∃x∀yGxy
51. ∀x∀y(Gxy → ∃zSyzx)
52. ∀x∀y(Gxy → ∃zPyzx)
♦53. ∀x∀y(Lxy → ∃zSxzy)
54. ∀x∀y∀z(Pxyz → ∃uSxuz)

55. ∀x∀y((Ex & Oy) → ∃z(Sxyz & Oz))
56. ∀x∀y((Ex & Oy) → ∃z(Sxyz & Ez))
57. ∀x∀y((Ex & Oy) → ∃z(Pxyz & Oz))
58. ∀x∀y((Ex & Oy) → ∃z(Pxyz & Ez))

PROBLEM 2 OF PROBLEM SET #2

(D) For each of the following *L* statements involving polyadic predicates, provide two interpretations: one under which the given statement is true and one under which the given statement is false.

Example: Given: ∀x∀y(Axy → Bxy)

1. 𝒟:	people	2. 𝒟:	people
Axy:	x is a parent of y	Axy:	x is the sister of y
Bxy:	y is a child of x	Bxy:	y is the brother of x

Under interpretation 1. the statement says 'Anybody who is the parent of another has that other as their child', which is true. Under interpretation 2. the statement says 'Anybody who is the sister of another has that other person as her brother', which is false.

59. Mab
60. ~Mab
61. Mab ∨ Mba
62. Mab & Mba
♦63. ∀xMax
64. ∃xMax & ~∀xMax
65. ∀x∀y(Lxy → Lyx)
66. ∀x∀y(Lxy → ~Lyx)
67. ∀x∃yAxy
68. ∃x∀yAxy
♦69. ∀x∀yAxy
70. ∃x∃yAxy & ∃z∃u~Azu
71. ∀x(Pxa → ∃yQxy)
72. ∃xAxa ∨ ∀yBay
73. Rabc
74. ∃xRabx
75. ∃xRaxb
76. ∃xRxbc

5.6 Symbolizing English I: Monadic Logic and Categorical Forms

Because *L* enables us to symbolize aspects of the logical structure of English statements that exist within atomic statements, *L* is a much more powerful tool for the analysis of arguments than is *SL*. The task of symbolizing English statements in *L* is, however, correspondingly more complex. We divide our discussion of symbolization into two sections. In this section we concentrate on monadic predicates. In Section 5.7 we turn to polyadic predicates and multiple quantifiers.

PROBLEM SET #3

Due: Monday 27 September at 11:00am in class

Problem 1 (1 point). Do problem 10.3 on p. 123 of BBJ.

Problem 2 (1 point). Do problem 10.9(a) on p. 124 of BBJ.

Problem 3 (1 point). Do problem 10.10(c) on p. 124 of BBJ.

Problem 4 (1 point). Do problem 10.14(b) on p. 124 of BBJ.

PROBLEM SET #4

Due: Monday 4 October at 11:00am in class

Problem 1 (2 points). Translate into logical notation the following argument, using the notation provided, and provide a derivation (using (R0)-(R9)) of the corresponding sequent: All entrants will win. There will be at most one winner. There is at least one entrant. Therefore, there is exactly one entrant. (*Ex*: *x* is an entrant; *Wx*: *x* is a winner.)

Problem 2 (2 points). Provide a three-rule proof procedure with the following properties: (a) it is complete but not sound, and (b) if any two of the three rules are removed, the resulting (one-rule) procedure is not complete. Explain *briefly* why the procedure has these properties.

PROBLEM SET #5

Due: Monday 18 October at 11:00am in class

Problem 1 (2 points). Prove that, if a sentence Q is true in all infinite models of a set of sentences Γ, then there is a natural number k such that Q is true in all finite models of Γ whose domain has k or more elements.

Problem 2 (2 points). Let the *spectrum* of a sentence be the set of all positive integers n such that the sentence has a finite model with a domain having exactly n elements. Give an example of a sentence whose spectrum is the set of all odd positive integers.

PROBLEM SET #6

Due: Monday 25 October at 11:00am in class

Problem 1 (2 points). Consider the following four sets of sentences: (a) The set whose members are all sentences of the language of PA that do not begin with a universal quantifier. (b) The set whose members are all and only those members of PA that do not contain the existential quantifier. (c) The set whose members are all and only those sentences of the language of PA that are false in at least one model of PA. (d) The set whose members are all and only those members of PA that contain at least three symbols (in official notation). For each of these sets, please indicate on the answer sheet whether the set is (1) a theory, (2) a consistent theory, (3) a decidable theory, (4) an axiomatizable theory, (5) a sufficiently powerful theory, and (6) a complete theory. (Assume that PA is consistent.)

Problem 2 (2 points). Let Γ be a set of interpretations of the language of PA, and let T be the set of sentences of the language of PA each of which is true in every member of Γ. Show that T is a theory.

PROBLEM SET #7

Due: Monday 1 November at 11:00am in class

Problem 1 (1 point). Find (as a product of prime numbers) the Gödel number of the expression '$\sim\exists x_{12}\forall x_{23}\sim(Sx_{11}=x_{27})$', using George and Velleman's system of Gödel numbering.

Problem 2 (1 point). Find the expression whose Gödel number (in George and Velleman's system of Gödel numbering) is: $1{,}162{,}213 \bullet 19^4 \bullet 957^{25} \bullet 1{,}190^7$.

Problem 3 (2 points). Suppose that T is an axiomatizable extension of PA. For any natural number n, say that T is consistent up to n exactly if there is no sentence P such that $T \vdash P$, $T \vdash \sim P$, and the Gödel numbers of P, $\sim P$, and proofs of P and $\sim P$ are all smaller than n. We can express this in the language of PA by defining $\text{ConUpTo}_T(x_5)$ to be the formula:

$\sim\exists x_1\exists x_2\exists x_3\exists x_4(x_1 < x_5 \,\&\, x_2 < x_5 \,\&\, x_3 < x_5 \,\&\, x_4 < x_5 \,\&\, \text{Proof}_T(x_1, x_2) \,\&\, \text{Proof}_T(x_3, x_4) \,\&\, \text{Neg}(x_1, x_3))$,

where $\text{Neg}(x_1, x_3)$ is a formula that represents $\{<n, m>\in\mathbf{N}^2$: n is the Gödel number of a sentence P and m is the Gödel number of $\sim P\}$. Then $\text{ConUpTo}_T(x_5)$ says that T is consistent up to x_5. Show that if T is consistent then, for every $n\in\mathbf{N}$, PA $\vdash \text{ConUpTo}_T(S^n0)$. (Use without proof the fact that, for every natural number n:

PA $\vdash \forall x_1(x_1 < S^n0 \rightarrow (x_1 = 0 \lor x_1 = S0 \lor x_1 = SS0 \lor \ldots \lor x_1 = S^{n-1}0)$.)

PROBLEM SET #8

Due: Monday 8 November at 11:00am in class

(Continuing Problem 3 of Problem Set #7; use the results of that problem)

Let T be a consistent and axiomatizable extension of PA, and let $\text{Proof}'_T(x_1, x_2)$ be the formula $\text{Proof}_T(x_1, x_2) \,\&\, \exists x_5(x_1 < x_5 \,\&\, x_2 < x_5 \,\&\, \text{ConUpTo}_T(x_5))$. Let $\text{Theorem}'_T(x_1)$ be the formula $\exists x_2\text{Proof}'_T(x_1, x_2)$, and let Con'_T be the sentence $\sim\exists x_1\exists x_2(\text{Theorem}'_T(x_1) \,\&\, \text{Theorem}'_T(x_2) \,\&\, \text{Neg}(x_1, x_2))$.

Problem 1 (2 points). Show that $\text{Proof}'_T(x_1, x_2)$ represents $\{<n, m>\in\mathbf{N}^2$: n is the Gödel number of a sentence and m is the Gödel number of a proof of that sentence in $T\}$.

Problem 2 (2 points). Prove Rosser's Theorem: if T is a consistent and axiomatizable extension of PA, then there is a sentence R_T such that neither $T \vdash R_T$ nor $T \vdash \sim R_T$. (Use without proof the result: $T \vdash \text{Con}'_T$.)

PROBLEM SET #9

Due: Monday 15 November at 11:00am in class

Problem 1 (2 points). Define a logician to be *accurate* iff everything she can prove is true; she never proves anything that is false. One day, an accurate logician visited the island of knights and knaves, in which each inhabitant is either a knight or a knave, and knights make only true statements and knaves make only false ones. The logician met two natives, A and B, who each made a statement such that from the two statements it follows that exactly one of the two natives must be a knight who is not provably so by the logician, but there is no way to tell which one it is. What two statements would work and why?

Problem 2 (2 points). Let T be a consistent and axiomatizable extension of PA. Show that $T \vdash \text{Con}'_T$, where Con'_T was defined in Problem Set #8. (Use without proof the *trichotomy law*:

PA $\vdash \forall x_1\forall x_2(x_1 < x_2 \lor x_2 < x_1 \lor x_1 = x_2)$.)