PRACTICE ARGUMENTS

I. CLASSICAL PROPOSITIONAL LOGIC

1. There will be nuclear war if and only if there is proliferation of nuclear weapons and unrest in the developing nations. Nuclear weapons will proliferate if and only if there is an increase in the use of nuclear power and nuclear safeguards are inadequate. There will be unrest in the developing nations if economic conditions do not improve. There will be an increase in the use of nuclear power and economic conditions will not improve. Therefore, nuclear war will be avoided only if there are adequate nuclear safeguards. (*N*, *W*, *U*, *P*, *S*, *E*)

2. I will find a job when I graduate only if I am well prepared, and I will be well prepared only if I can read and write extremely well and have a good technical education. I will read and write extremely well if and only if I take a lot of Humanities courses. But if I take a lot of Humanities courses, I will not take many technical courses, and if I don't take many technical courses, then I won't have a good technical education. Therefore, I won't find a job when I graduate. (*J*, *W*, *R*, *E*, *H*, *T*)

3. If the Monetarists are right, then there is an increase in inflation if and only if the money supply increases too fast. If the Keynesians are right, then there is an increase in inflation if and only if there is a decrease in unemployment. If the Libertarians are right, then there is an increase in inflation if and only if the federal government spends more than it takes in. The federal government spends more than it takes in only if taxes are too low. There is no decrease in unemployment and taxes are not too low, but there is inflation. Therefore, neither the Monetarists, nor the Keynesians, nor the Libertarians are right. (*M*, *I*, *S*, *K*, *D*, *L*, *F*, *T*)

4. Add to the premises of Argument 3: The money supply increases too fast only if taxes are too low.

II. PROPOSITIONAL MODAL LOGIC

5. It is possible that not all living things are able to feel pain. For, necessarily, all living things are able to feel pain only if all living things have nervous systems. And, necessarily, if plants do not have nervous systems, then not all living things have nervous systems. And it is possible that plants do not have nervous systems. (*L*, *N*, *P*)

6. The reduction of violence is necessary and sufficient for making drugs legal. But it is possible that more people will use drugs if drugs are made legal. And, necessarily, violence will not be reduced if more people use drugs. It follows that drugs will not necessarily be made legal. (*V*, *L*, *P*)

7. I can know nothing. For every piece of reasoning must start somewhere. And, necessarily, if every piece of reasoning starts somewhere, then every piece of reasoning begins with an unsupported premise. And, necessarily, if every piece of reasoning begins with an unsupported premise, then I know nothing. (*K*: I know nothing; *S*: every piece of reasoning starts somewhere; *U*: every piece of reasoning begins with an unsupported premise.)

8. It is not necessary that ghosts fail to exist. Therefore, the conjunction of "necessarily, spirits exist" and "necessarily, ghosts don't exist" entails "necessarily, God exists". (*A*: spirits exist; *B*: ghosts exist; *C*: God exists.)

9. If your argument is valid, I must be mistaken. If your argument is valid, it must be necessarily valid. So if your argument is valid, I must be necessarily mistaken. (*V*: your argument is valid; *M*: I am mistaken.)

10. Necessarily, it is possible for you to walk to the door only if it is possible for you to walk to the halfway point between yourself and the door. Necessarily, however, it is possible for you to walk to the halfway point between yourself and the door only if it is possible for you to walk to a point *halfway* to the halfway point. Necessarily, if it is impossible for you to walk halfway to the halfway point only if it is

impossible for you to walk to the door, then it is possible for you to walk to the door only if it is possible for you to perform an infinite number of acts in a finite period of time. But it is necessarily impossible for you to perform an infinite number of acts in a finite period of time. So it is necessarily impossible for you to walk to the door. (*D*: you walk to the door; *H*: you walk to the halfway point between yourself and the door; *P*: you walk halfway to the halfway point; *F*: you perform an infinite number of acts a finite period of time.)

11. Necessarily, if the workplace is a meritocracy, then it is necessary that Joe will be hired if Joe is the most qualified candidate. But, necessarily, if networking plays a role in who gets jobs, then it is possible that Joe will not be hired despite being the most qualified candidate. Furthermore, it is possible that networking does play a role in who gets gobs. Therefore, it is possible that the workplace is not a meritocracy. (*W*: the workplace is a meritocracy; *H*: Joe will be hired; *Q*: Joe is the most qualified candidate; *N*: networking plays a role in who gets jobs.)

12. Necessarily, it could not have failed to be the case that either it rains or it doesn't rain. For, necessarily, if it could have failed to be the case that either it rains or it doesn't rain, then the argument from the claim that the sky is blue to the claim that either it rains or it doesn't rain would not have been valid. And that argument is necessarily valid. (*R*: either it rains or it doesn't rain; *Q*: the sky is blue.)

13. Necessarily, being possibly undetermined is a necessary but not a sufficient condition for human behavior's being free. Necessarily, the laws of subatomic physics are statistical only if it is necessary that human behavior is necessarily undetermined. And the laws of subatomic physics are necessarily statistical. It follows that human behavior is necessarily free. (*D*: human behavior is determined; *F*: human behavior is free; *L*: the laws of subatomic physics are statistical.)

14. Necessarily, if God's existence is contingent, then God's existence is a matter of metaphysical luck. But it is impossible for God's existence to be a matter of metaphysical luck. Necessarily, God's existence is not impossible if it is possible for an omnipotent and perfectly good being to exist. And it is necessarily possible for an omnipotent and perfectly good being to exist. Therefore, God's existence is necessary. (*G*: God exists; *M*: God's existence is a matter of metaphysical luck; *O*: an omnipotent and perfectly good being exists.)

15. Necessarily, if it is possible that God believes on Monday that I will lie on Tuesday, then either it is possible for me to make one of God's past beliefs false, or it is impossible that I refrain from lying on Tuesday. Necessarily, it is impossible for me to make one of God's past beliefs false if either God is necessarily infallible or it is impossible for me to change the past. It is impossible for me to change the past. It follows that, necessarily, it is possible that if God necessarily believes on Monday that I will lie on Tuesday then it is impossible that I refrain from lying on Tuesday. (*B*: God believes on Monday that I will lie on Tuesday; *F*: I make one of God's past beliefs false; *R*: I refrain from lying on Tuesday; *I*: God is infallible; *P*: I change the past.)

16. Necessarily, if time is real then it is impossible for God to be outside of time. For, as St. Thomas Aquinas pointed out, necessarily, if it is possible for God to be outside of time, then it is possible for God to see all of time (past, present, and future) at a glance. But, necessarily, if God sees all of time at a glance, then the future already exists. And, necessarily, if the future already exists, then I have already committed sins that I will commit in the future. But it is definitely possible that, necessarily, both time is real and I have *not* already committed sins that I will commit in the future. (*T*: time is real; *O*: God is outside of time; *S*: God sees all of time at a glance; *F*: the future already exists; *I*: I have already committed sins that I will commit in the future.

17. Necessarily, there is life after death if and only if God necessarily exists. For, necessarily, either God necessarily exists or only matter exists. Necessarily, if only matter exists, there is no life after death. Necessarily, God exists if and only if God is both perfectly good and omnipotent. Necessarily, if God is omnipotent, then it is possible for God to raise humans from the dead. Necessarily, if God is perfectly good, then God wants to raise humans from the dead if human resurrection is necessary for human fulfillment. Human resurrection is indeed necessary for human fulfillment. Necessarily, if God wants to

raise humans from the dead and it is possible for God to do so, then there is life after death. (*L*: there is life after death; *G*: God exists; *M*: only matter exists; *P*: God is perfectly good; *O*: God is omnipotent; *D*: God raises humans from the dead; *W*: God wants to raise humans from the dead; *R*: human resurrection occurs; *F*: human fulfillment occurs.)

18. Necessarily, if time travel is possible, then it is possible for me to go back in time and point a loaded gun at my infant self. Necessarily, if it is possible for me to go back in time and point a gun at my infant self, then it is possible for me to kill my infant self. Necessarily, if it is possible for me to kill my infant self, then it is possible for me to change the past. But it is impossible for me to change the past. It follows that time travel is necessarily impossible. (*T*: time travel occurs; *G*: I go back in time and point a loaded gun at my infant self; *K*: I kill my infant self; *P*: I change the past.)

19. Necessarily, if time travel is possible, then it is possible for me to go back in time and point a loaded gun at my infant self. Necessarily, if I go back in time and point a loaded gun at my infant self, then my killing him is compossible with the fact that he is within the range of my gun. Necessarily, if I kill my infant self, then I exist at the same time as my infant self. It is not possible for me to exist at the same time as my infant self without being a later stage of my infant self. Necessarily, if I kill my infant self and I am a later stage of my infant self, then resurrection occurs. But resurrection is impossible. It follows that time travel is necessarily impossible. (*T*: time travel occurs; *G*: I go back in time and point a loaded gun at my infant self; *K*: I kill my infant self; *F*: my infant self is within the range of my loaded gun; *C*: I exist at the same time as my infant self; *L*: I am a later stage of my infant self; *R*: resurrection occurs.)

20. Necessarily, I am an *essentially simple* object exactly if it is necessary that if I exist then I have no proper spatial part. Necessarily, if time travel is possible, then it is also possible that I visit my younger self. Necessarily, if I visit my younger self, then I exist and I have a proper spatial part. It follows that my being an essentially simple object and the possibility of time travel are not compossible. (*S*: I am an essentially simple object; *E*: I exist; *P*: I have a proper spatial part; *V*: I visit my younger self; *T*: time travel occurs.)

21. It is necessarily possible that, if God necessarily exists, then resurrection is necessarily possible. It follows that, if God necessarily exists, then resurrection is necessarily possible. (*G*: God exists; *R*: resurrection occurs.)

22. The Gödel sentence of Peano Arithmetic (PA) is necessary. Therefore, necessarily, if the Gödel sentence of PA is necessary, then the Gödel sentence of PA is necessarily equivalent to "the Gödel sentence of PA is necessary". (*G*: the Gödel sentence of PA.)

23. I know that: the Gödel sentence of Peano Arithmetic (PA) is true exactly if I don't know that the Gödel sentence of PA is true. I also know that, if I know that the Gödel sentence of PA is true, then the Gödel sentence of PA is true. It follows that I know that the Gödel sentence of PA is true. (*G*: the Gödel sentence of PA.)

24. It is provable in Peano Arithmetric (PA) that the Gödel sentence of PA is materially equivalent to "the Gödel sentence of PA is not provable in PA". Therefore, if it is not provable in PA that it is both raining and not raining, then it is not provable in PA that it is not provable in PA that it is both raining and not raining—which is essentially Gödel's second incompleteness theorem. (*G*: the Gödel sentence of PA; *R*: it is raining.)

III. CLASSICAL QUANTIFIED LOGIC

25. If there are any barbers, then there is a barber who shaves all and only those who do not shave themselves. Therefore, there are no barbers. (*Bx*: *x* is a barber; *Sxy*: *x* shaves *y*.)

26. Every person is honest with any person who is honest with every person. There is at least one person who is honest with every person. Therefore, every person is honest with at least one person who is honest with them. (*Px*: *x* is a person; *Hxy*: *x* is honest with *y*.)

27. Every student who cheated bribed at least one professor. Some students were accused and so was every professor they bribed. All accused students cheated. Therefore, some professor cheated. (*Sx*: *x* is a student; *Px*: *x* is a professor; *Ax*: *x* was accused; *Cx*: *x* cheated; *Bxy*: *x* bribed *y*.)

28. Replace the conclusion of Argument 27 with: Some professor cheated or is not a student.

29. There is someone who teaches everyone who is taught by anyone. There is someone who teaches everyone who teaches anyone. Everyone who does not teach anyone is taught by someone. Therefore, there is someone who teaches everyone. (*Txy*: *x* teaches *y*.)

IV. QUANTIFIED MODAL LOGIC

30. Necessarily, everyone is possibly a mathematician. It is possible that someone is necessarily rational. Therefore, it is possible that someone is possibly a rational mathematician. (*Mx: x* is a mathematician; *Rx*: *x* is rational.)

31. It is possible that everything is physical. Necessarily, something is mental. Therefore, it is possible that something mental is physical. (*Px*: *x* is physical; *Mx*: *x* is mental.)

32. Necessarily, everything is either mental or physical. Therefore, it is possible that everything is mental, or it is possible that something is physical. (*Mx*: *x* is mental; *Px*: *x* is physical.)

33. Everything is either necessarily material or necessarily non-material. Therefore, necessarily, God is material exactly if God is necessarily material. (*Mx*: *x* is material; *g*: God.)

34. Necessarily, some number is even. Every number is possibly thought of by God. Therefore, it is possible that some even number is thought of by God. (*Nx*: *x* is a number; *Ex*: *x* is even; *Tx*: *x* is thought of by God.)

35. Necessarily, every event such that something material possibly causes it is determined. Necessarily, some event that is possibly determined is not possibly free. Therefore, some event such that everything material necessarily does not cause it is not free. (*Ex*: *x* is an event; *Mx*: *x* is material; *Dx*: *x* is determined; *Fx*: *x* is free; *Cxy*: *x* causes *y*.)

36. Necessarily, every contingent being is possibly causally dependent on some being. Necessarily, no necessary being is possibly causally dependent on anything. Necessarily, every physical entity is contingent. It follows that, necessarily, no physical entity is a necessary being. (*Px*: *x* is physical; *Dxy*: *x* is causally dependent on *y*.)

37. Necessarily, no god possibly causes any possibly free act that she wants to occur. Necessarily, no god possibly causes any possibly free act that she does not want to occur. Therefore, necessarily, no god possibly causes any necessarily free act. (*Gx*: *x* is a god; *Fx*: *x* is a free act; *Cxy*: *x* causes *y*; *Wxy*: *x* wants *y* to occur.)

38. Necessarily, every act is possibly caused by a desire. Necessarily, every desire is possibly caused by a brain process. Necessarily, for all *x*, *y*, and *z*, if *x* is possibly caused by *y* and *y* is possibly caused by *z*, then *x* is possibly caused by *z*. It follows that, necessarily, every act is possibly caused by a brain process. (*Ax*: *x* is an act; *Dx*: *x* is a desire; *Bx*: *x* is a brain process; *Cxy*: *x* causes *y*.)

39. Necessarily, every existing object is such that, necessarily, it is material if God is not material. God is not material, and neither is the number zero. Every (existing or non-existing) object possibly exists. Therefore, necessarily, every existing object is such that, possibly, it is material and the number zero is not material. (*Mx*: *x* is material; *g*: God; *n*: the number zero.)

40. Necessarily, everyone is either righteous or wicked. Necessarily, no one suffers any injustice from anyone who is righteous. Necessarily, one suffers an injustice from someone only if that someone takes away something possibly good from one. Necessarily, only virtuous things are possibly good. Necessarily, nothing virtuous is taken away by anyone from anyone who is righteous. Necessarily, everyone who is wicked possesses nothing that is possibly good. Necessarily, one takes away something from someone only if that someone possesses the thing. It follows that, necessarily, no one suffers any

injustice from anyone. (*Rx*: *x* is righteous; *Wx*: *x* is wicked; *Ix*: *x* is an injustice; *Gx*: *x* is good; *Vx*: *x* is virtuous; *Pxy*: *x* possesses *y*; *Sxyz*: *x* suffers *y* from *z*; *Txyz*: *x* takes *y* away from *z*.)

41. Necessarily, something is such that everything possibly depends on it. Necessarily, anything that possibly depends on a thing is possibly sustained by it. Necessarily, anything that possibly sustains everything possibly depends on everything. It follows that, possibly, something possibly sustains something and is possibly sustained by it. (*Dxy*: *x* depends on *y*; *Sxy*: *x* sustains *y*.)

42. It is possible that some existing deity is such that, necessarily, it exists, it is a deity, and it causes every existing material object. Therefore, it is necessary that every existing material object is caused by some existing deity (or other). (*Dx*: *x* is a deity; *Mx*: *x* is a material object; *Cxy*: *x* causes *y*.)

43. Some natural number is such that every natural number is necessarily larger than it. Every natural number is necessarily a natural number. Therefore, given any natural number, it is necessary that it is larger than some natural number. (*Nx*: *x* is a natural number; *Lxy*: *x* is larger than *y*.)

44. Necessarily, every proposition has a negation which is a proposition. Necessarily, for any proposition and any negation of it, it is necessary that: the negation is true if and only if the proposition is not true. Necessarily, for any proposition, and for any object with respect to which the proposition is singular, it is necessary that: the proposition is not true if the object does not exist. Necessarily, for any proposition, for any object with respect to which the proposition is singular, and for any negation of the proposition, it is necessary that: the negation of the proposition does not exist if the object does not exist. Necessarily, some proposition is singular with respect to a contingent object. It follows that, necessarily, some proposition is such that it is possible for it to be true without existing (*Px*: *x* is a proposition; *Ox*: *x* is an object; *Tx*: *x* is true; *Nxy*: *x* is a negation of *y*; *Sxy*: *x* is singular with respect to *y*.)

45. Necessarily, a first proposition entails a second one exactly if the first is a subset of the second. Necessarily, if A is a subset of B, then, necessarily, if B exists then A exists. Necessarily, for any proposition, and for any object with respect to which the proposition is singular, it is necessary that if the proposition exists then the object exists. Necessarily, the proposition that someone is wise is entailed by the proposition that Socrates is wise, which is singular with respect to the object Socrates. It follows that it is necessarily impossible for the proposition that someone is wise to exist without Socrates. (*Px*: *x* is a proposition; *Ox*: *x* is an object; *Exy*: *x* entails *y*; *Bxy*: *x* is a subset of *y*; *Sxy*: *x* is singular with respect to *y*; *q*: the proposition that someone is wise; *r*: the proposition that Socrates is wise; *s*: Socrates.)

46. Necessarily, for any set, it is necessary that: the set exists if and only if all of its members exist. Necessarily, for any set, it is necessary that every member of the set is necessarily a member of the set. Necessarily, a first set is subset of a second one if and only if every member of the first is a member of the second. It follows that, necessarily, if a first set is a subset of a second, then, necessarily, the second exists only if the first exists. (*Sx*: *x* is a set; *Mxy*: *x* is a member of *y*; *Bxy*: *x* is a subset of *y*.)

47. Necessarily, for any propositions A, B, and C, A and B *commitment-entail* C exactly if, necessarily, given any person, if A is possible and follows from some proposition that the person believes and B follows from some proposition that the person desires, then C follows from some proposition that the person desires. Necessarily, for any propositions A and B, necessarily, B follows from A exactly if, necessarily, B is true if A is. Therefore, if from a proposition A a proposition follows which, necessarily, is true exactly if the proposition C follows from the proposition B, then A and B commitment-entail C. (*Px*: *x* is a proposition; *Rx*: *x* is a person; *Tx*: *x* is true; *Bxy*: *x* believes *y*; *Dxy*: *x* desires *y*; *Fxy*: *x* follows from *y; Cxyz: x* and *y* commitment-entail *z*.)

48. Necessarily, there is an argument such that, necessarily, the argument is valid exactly if all of its premises are true and the argument is invalid exactly if a conclusion of it is true. Necessarily, an argument is valid exactly if, necessarily, if all of its premises are true, then a conclusion of it is true. Necessarily, for any argument, it is necessary that, if the argument is valid, then it is necessarily valid. It follows that, necessarily, there is an argument such that it is valid exactly if it is invalid. (*Ax*: *x* is an argument; *Vx*: *x* is valid; *Tx*: *x* is true; *Pxy*: *x* is a premise of *y*; *Cxy*: *x* is a conclusion of *y*.)

49. It is possible that some existing object is such that, necessarily, no existing object is greater than it. Necessarily, every possible object that exists is greater than any possible object that does not exist. Something exists. It follows that some existing object is such that, necessarily, no existing object is greater than it. (*Gxy*: *x* is greater than *y*.)

V. IDENTITY

50. Some existing object is necessarily identical with God. Every existing object is necessarily created. It follows that God is necessarily created. (*Cx*: *x* is created; *g*: God.)

51. Necessarily, some existing object is identical with God. Therefore, some existing object is necessarily identical with God. (*g*: God.)

52. Necessarily, if a first prescription follows from a second one, then, necessarily, if a balance of reasons supports the second prescription, then a balance of reasons supports the first prescription. Necessarily, at most one balance of reasons exists. It follows that, necessarily, if a first prescription follows from a second one, then, necessarily, every balance of reasons that supports the second prescription also supports the first prescription. (*Px*: *x* is a prescription; *Bx*: *x* is a balance of reasons; *Sxy*: *x* supports *y*; *Fxy*: *x*: follows from *y*.)

53. It is possible that God is immaterial. Therefore, it is possible that something identical with God is immaterial. (*Ix*: *x* is immaterial; *g*: God.)

54. Every existing object is such that, necessarily, it is identical to some existing object. It is possible that some existing object is immaterial. It follows that some existing object is possibly immaterial. (*Ix*: *x* is immaterial.)

55. Bucephalus might have sired a horse other than every horse he actually sired. Therefore, it is not essential to everything that is not actually a horse sired by Bucephalus that it not be a horse sired by Bucephalus. (*Hx*: *x* is a horse; *Sxy*: *x* sires *y*; *b*: Bucephalus.)

56. Bucephalus might have sired a horse other than every horse he actually sired. It is essential to every actual object that is not actually a horse sired by Bucephalus that it not be a horse sired by Bucephalus. Therefore, there might have existed an object distinct from every actual object. (*Hx*: *x* is a horse; *Sxy*: *x* sires *y*; *b*: Bucephalus.)

57. Necessarily, if a wooden table *x* might have been the only table originally formed from a hunk of matter *y*, then it is impossible that some wooden table distinct from *x* is the only table formed from *y*. Necessarily, if a wooden table *x* is the only table originally formed from a hunk of matter *y*, and *w* is any hunk of matter that is sufficiently similar to *y*, then *x* might have been the only table originally formed from *w*. Necessarily, if a wooden table *x* is the only table originally formed from a hunk of matter *y*, and *z* is any hunk of matter that is not sufficiently similar to *y*, then *x* could not have been the only table originally formed from *z*. Therefore, it is not the case that some wooden table is the only table originally formed from a hunk of matter *h* such that some hunk of matter *h″* is sufficiently similar to *h*, some hunk of matter *h′* is sufficiently similar to *h″* but not sufficiently similar to *h*, and it is possible that some wooden table is the only table originally formed from *h′*. (*Wx*: *x* is a wooden table; *Mx*: *x* is a hunk of matter; *Fxy*: *x* is the only table originally formed from *y*; *Sxy*: *x* is sufficiently similar to *y*.)

58. It is possible that some existing object is such that possibly some existing object is distinct from it. God is omnipotent. Necessarily, every existing object is such that, necessarily, it exists if God is omnipotent. It follows that at least two objects exist. (*Ox*: *x* is omnipotent; *g*: God.)

59. Necessarily, every human is such that necessarily someone loves her. Necessarily, for every *x*, *y*, and *z*, if it is possible that *y* loves *x* and it is possible that *z* loves *x*, then *y* is identical to *z*. It follows that, necessarily, every human is such that someone necessarily loves her. (*Hx*: *x* is human; *Lxy*: *y* loves *x*.)

60. Necessarily, a reason is *complete* exactly if, necessarily, for any prescription, if the reason supports the prescription, then the reason could not have been a reason without supporting the prescription.

Necessarily, every complete reason could have been the *only* complete reason. Necessarily, "come and talk" and "talk" are prescriptions. Necessarily, if some complete reason supports "come and talk", then some complete reason supports "talk". It follows that, necessarily, every complete reason that supports "come and talk" also supports "talk". (*Rx*: *x* is a reason; *Px*: *x* is a prescription; *Cx*: *x* is complete*; Sxy: x* supports *y*; *a*: "come and talk"; *b*: "talk".)

VI. PREDICATE ABSTRACTION

61. Necessarily, the supreme commander of the U.S. armed forces is the most powerful person in the world (*de dicto*). The U.S. President might not have been the supreme commander of the U.S. armed forces (*de re*). Therefore, the U.S. President might not have been the most powerful person in the world (*de re*). (π: the U.S. President; α: the supreme commander of the U.S. armed forces; β: the most powerful person in the world.)

62.The U.S. President is necessarily human (*de re*). The U.S. President could have been the author of *Jane Eyre* (*de re*). Therefore, it is possible that the author of *Jane Eyre* is human (*de dicto*). (*Hx*: *x* is human; π: the U.S. President; α: the author of *Jane Eyre*.)

63. The teacher of Alexander is possibly distinct from the most famous Greek philosopher (*de re*). Aristotle is the teacher of Alexander. Therefore, the most famous Greek philosopher is possibly distinct from Aristotle (*de re*). (*a*: Aristotle; β: the most famous Greek philosopher; γ: the teacher of Alexander.)

64. The U.S. President might have been the French President (double *de re*). Therefore, The U.S. President is the French President. (π: the U.S. President; ϕ: the French President.)

65. It is impossible for different women to give birth to the same individual. The author of *Jane Eyre* and the author of *Wuthering Heights* might have been distinct women (*de dicto*). Necessarily, the author of *Jane Eyre* has the property (*de re*) of necessarily giving birth to the French ambassador. Necessarily, the author of *Wuthering Heights* has the property (*de re*) of necessarily giving birth to the German ambassador. Therefore, the French ambassador and the German ambassador might have been distinct (*de dicto*). (*Wx*: *x* is a woman; *Bxy*: *x* gives birth to *y*; α: the author of *Jane Eyre*; β: the author of *Wuthering Heights*; γ: the French ambassador; δ: the German ambassador.)

66. Necessarily, the Grand Inquisitor has the property (*de re*) of being possibly a Catholic feared by every Catholic. Possibly, the Grand Inquisitor has the property (*de re*) of being necessarily such that the only Catholic he fears is the Pope. Therefore, it is possible that the Grand Inquisitor has the property (*de re*) of being not necessarily distinct from the Pope. (*Cx*: *x* is a Catholic; *Fxy*: *x* fears *y;* γ: the Grand Inquisitor; π: the Pope.)

67. Necessarily, the Grand Inquisitor has the property *(de re*) of being possibly a Catholic feared by every Catholic. Possibly, the Pope has the property (*de re*) of being necessarily the only Catholic that the Grand Inquisitor fears. Possibly, the Grand Inquisitor is necessarily identical with Charles XVI (*de dicto*). Possibly, the Pope is necessarily identical with Paul XXII (*de dicto*). Therefore, Charles XVI is identical with Paul XXII. (*Cx*: *x* is a Catholic; *Fxy*: *x* fears *y*; γ: the Grand Inquisitor; π: the Pope; *c*: Charles XVI; *p*: Paul XXII.)

VII. DEFINITE DESCRIPTIONS

68. The tallest golden mountain exists, and every existing object could have failed to be a tallest golden mountain. It follows that the tallest golden mountain has the property (*de re*) of being such that, possibly, it is not the case that it is identical to the tallest golden mountain. (*Gx*: *x* is a golden mountain; *Txy*: *x* is at least as tall as *y*.)

69. The possible perfect being is self-identical. The perfect being has the property (*de re*) of being necessarily the perfect being. It follows that the possible perfect being is the perfect being. (*Px*: *x* is a perfect being.)

70. Every tallest existing giraffe is a tallest existing animal. There is a unique tallest existing giraffe, and there is a unique tallest existing animal. Therefore, the tallest existing giraffe is the tallest existing animal. (*Gx*: *x* is a giraffe; *Ax*: *x* is an animal; *Txy*: *x* is at least as tall as *y*.)

71. Necessarily, the most powerful being exists. Therefore, it is not the case that, for every existing object, the most powerful being has the property (*de re*) of being possibly distinct from that object. (*Pxy*: *x* is at least as powerful as *y*.)

72. Necessarily, the most powerful being exists. The possible most powerful being exists. Therefore, the possible most powerful being has the property (*de re*) of being possibly the most powerful being. (*Pxy*: *x* is at least as powerful as *y*.)

73. Necessarily, either the fountain of youth has the property of existence or the fountain of youth has the property of non-existence. Therefore, every existing object identical with the possible fountain of youth is such that possibly it is identical with the fountain of youth. (*Fx*: *x* is a fountain of youth.)

74. It is possible that the creator of the largest solar system is the most powerful person (*de dicto*). Necessarily, the creator of the largest solar system creates only material entities. Therefore, it is possible that the most powerful person creates a solar system that is a material entity. (*Sx*: *x* is a solar system; *Rx*: *x* is a person; *Mx*: *x* is a material entity; *Lxy*: *x* is at least as large as *y*; *Pxy*: *x* is at least as powerful as *y*; *Cxy*: *x* is creates *y*.)

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(From which the above arguments were taken, usually with considerable modifications. The original authors need not endorse the modified arguments. Please do not quote or circulate without permission from the original authors.)

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REVIEW OF CLASSICAL PROPOSITIONAL LOGIC: NATURAL DEDUCTION

I. LOGICAL CONNECTIVES

p	q	$\sim p$	$p \,\&\, q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

II. TRANSLATIONS

p but q	$p \,\&\, q$
Neither p nor q	$\sim p \,\&\, \sim q$
Either p or q	$p \vee q$
p unless q	$p \vee q$
Assuming p, q	$p \rightarrow q$
q if p	$p \rightarrow q$
p only if q	$p \rightarrow q$
p exactly if q	$p \leftrightarrow q$

III. REPLACEMENT RULES

Name	Abbrev.	Rule
Double Negation	DN	$\sim\sim p \Leftrightarrow p$
Tautology	Taut	$p \Leftrightarrow (p \,\&\, p)$ $p \Leftrightarrow (p \vee p)$
Commutation	Comm	$(p \,\&\, q) \Leftrightarrow (q \,\&\, p)$ $(p \vee q) \Leftrightarrow (q \vee p)$
Association	Assoc	$((p \,\&\, q) \,\&\, r) \Leftrightarrow (p \,\&\, (q \,\&\, r))$ $((p \vee q) \vee r) \Leftrightarrow (p \vee (q \vee r))$
Distribution	Dist	$(p \,\&\, (q \vee r)) \Leftrightarrow ((p \,\&\, q) \vee (p \,\&\, r))$ $(p \vee (q \,\&\, r)) \Leftrightarrow ((p \vee q) \,\&\, (p \vee r))$
De Morgan's Law	DeM	$\sim(p \vee q) \Leftrightarrow (\sim p \,\&\, \sim q)$ $\sim(p \,\&\, q) \Leftrightarrow (\sim p \vee \sim q)$
Material Implication	Impl	$(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$
Transposition	Trans	$(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p)$
Absorption	Abs	$(p \rightarrow q) \Leftrightarrow (p \rightarrow (p \,\&\, q))$
Exportation	Exp	$(p \rightarrow (q \rightarrow r)) \Leftrightarrow ((p \,\&\, q) \rightarrow r)$
Negated Conditional	NC	$\sim(p \rightarrow q) \Leftrightarrow (p \,\&\, \sim q)$
Material Equivalence	Equiv	$(p \leftrightarrow q) \Leftrightarrow ((p \rightarrow q) \,\&\, (q \rightarrow p))$ $(p \leftrightarrow q) \Leftrightarrow ((p \,\&\, q) \vee (\sim p \,\&\, \sim q))$

IV. WHOLE-LINE INFERENCE RULES

Name	Abbrev.	Rule
Conjunction Elimination	CE	$\dfrac{p \,\&\, q}{p}$ $\quad$ $\dfrac{p \,\&\, q}{q}$
Conjunction Introduction	CI	$\dfrac{\begin{array}{l}p\\ q\end{array}}{p \,\&\, q}$
Modus Ponens	MP	$\dfrac{\begin{array}{l}p \rightarrow q\\ p\end{array}}{q}$
Modus Tollens	MT	$\dfrac{\begin{array}{l}p \rightarrow q\\ {\sim}q\end{array}}{{\sim}p}$
Hypothetical Syllogism	HS	$\dfrac{\begin{array}{l}p \rightarrow q\\ q \rightarrow r\end{array}}{p \rightarrow r}$
Disjunctive Syllogism	DS	$\dfrac{\begin{array}{l}p \vee q\\ {\sim}p\end{array}}{q}$ $\quad$ $\dfrac{\begin{array}{l}p \vee q\\ {\sim}q\end{array}}{p}$
Disjunctive Addition	DA	$\dfrac{p}{p \vee q}$ $\quad$ $\dfrac{p}{q \vee p}$
Constructive Dilemma	CD	$\dfrac{\begin{array}{l}p \rightarrow q\\ r \rightarrow s\\ p \vee r\end{array}}{q \vee s}$

V. CONDITIONAL PROOF (CP) AND REDUCTIO (RAA)

1. Any sentence whatsoever may be *introduced* as an assumption, with justification 'PA-CP' or 'PA-RAA'. (Introducing an assumption *starts* a *subproof*.)

2. Every assumption must be *discharged* after it is introduced. (The subproof that started when the assumption was introduced *ends* on the line immediately before the assumption is discharged.)

   (1) A CP assumption is discharged at a line of the form $p \rightarrow q$, where p is the assumption and q is the line immediately before the assumption is discharged.

   (2) An RAA assumption is discharged at a line of the form ${\sim}p$, if the assumption is p, or p, if the assumption is ${\sim}p$; the line immediately before the assumption is discharged must be of the form $q \,\&\, {\sim}q$.

3. If a second assumption is introduced after a first assumption but before the first assumption is discharged, then the second assumption must be discharged before the first assumption is discharged. (So if two subproofs overlap, then one of them must be nested inside the other.)

4. After an assumption is discharged, the lines from the introduction of the assumption to the line immediately before the assumption is discharged may no longer be used.

PETER B. M. VRANAS

REVIEW OF CLASSICAL PROPOSITIONAL LOGIC: SEMANTIC TABLEAUX (TRUTH TREES)

I. THE STEPS OF THE METHOD

Step 1: List vertically the premises and the *negation* of the conclusion; this is the *trunk* of the tree.

Step 2: Try to find an *interpretation* (i.e., an assignment of truth values to the sentence letters) on which all sentences in the trunk are true. To do this, successively check the *complex* sentences (i.e., those that consist neither of a single sentence letter nor of the negation of a single sentence letter) in the trunk and decompose them into simpler sentences by applying the following *whole-line* decomposition rules:

$p \& q$
p q

$p \vee q$	
p	q

$p \to q$	
$\sim p$	q

$p \leftrightarrow q$	
p q	$\sim p$ $\sim q$

$\sim(p \& q)$	
$\sim p$	$\sim q$

$\sim(p \vee q)$
$\sim p$ $\sim q$

$\sim(p \to q)$
p $\sim q$

$\sim(p \leftrightarrow q)$	
p $\sim q$	$\sim p$ q

The Double Negation replacement rule may also be used to eliminate double tildes.

Step 3: Stop if and only if:

(1) Either every branch is *closed* (i.e., it contains, for some—maybe complex—sentence *p*, both *p* and ~*p*), in which case the sentences in the trunk are inconsistent and the argument is *valid*;

(2) Or at least one *open* (i.e., not closed) branch is *finished* (i.e., all complex sentences in the branch have been checked), in which case the sentences in the trunk are consistent and the argument is *invalid*.

(Notation: Put 'x' below every closed branch and 'o' below every finished open branch.)

Step 4: If the argument is invalid, a finished open branch provides a *countermodel* of the argument, namely an interpretation on which all premises are true and the conclusion is false: it is an interpretation which assigns 'true' to every sentence letter that occurs alone in a line of the branch, and assigns 'false' to every sentence letter whose negation occurs alone in a line of the branch. (You may assign 'true' or 'false' at your choice to any other sentence letter.)

II. RULES OF THUMB

1. Apply non-branching rules before branching rules.
2. Decompose first sentences that result in closed branches.
3. If the previous two rules do not apply, decompose first more complex sentences.
4. Once a branch is closed, do not decompose any complex sentence in it.
5. Once you find a finished open branch, stop: do not examine other branches.

INTRODUCTION TO MODAL LOGIC

I. MOTIVATION

Some obviously valid arguments cannot be proven to be valid in classical logic. For example: "It is impossible for you to run faster than light. Therefore, you will not run faster than light."

II. MODAL EXPRESSIONS

1. Modal *predicates* ('necessary', 'possible', 'impossible', 'contingent', 'non-contingent') can be distinguished from modal *adverbs* ('necessarily', 'possibly',...), but we will use interchangeably, e.g., "necessarily, God exists", "it is necessary that God exists", and "'God exists' is necessary".

2. The sentence operators '$\Box$' (or 'L') and '$\Diamond$' (or 'M') stand for 'necessarily' and 'possibly' respectively, and can be prefixed to any sentence to form a sentence. So they can also be *iterated*: '$\Box\Diamond A$' stands for "necessarily, A is possible".

3. Necessity and possibility are interdefinable: $\Box A \Leftrightarrow \sim\Diamond\sim A$ and $\Diamond A \Leftrightarrow \sim\Box\sim A$. All modal operators can be defined in terms of possibility (or necessity):

A is possible	$\Diamond A$ (equivalently: $\sim\Box\sim A$)
A is necessary	$\sim\Diamond\sim A$ (equivalently: $\Box A$)
A is impossible	$\sim\Diamond A$ (equivalently: $\Box\sim A$)
A is contingent	$\Diamond A$ & $\Diamond\sim A$ (equivalently: $\sim(\Box\sim A \vee \Box A)$)
A is non-contingent	$\sim(\Diamond A$ & $\Diamond\sim A)$ (equivalently: $\Box\sim A \vee \Box A$)

4. Modal operators are *not* truth-functional. For example: some true sentences are necessary ("2+2 = 4") and some true sentences are not necessary ("I am a philosopher").

5. *If* it is assumed that every necessary sentence is true (some systems of modal logic do *not* assume this), then sentences can be classified as follows:

	True	False
Non-contingent	Necessary (thus possible)	Impossible
Contingent (thus possible)	Contingently true	Contingently false

III. REPLACEMENT RULE

Name	Abbrev.	Rule
Modal Negation	MN	$\sim\Box p \Leftrightarrow \Diamond\sim p$ $\sim\Diamond p \Leftrightarrow \Box\sim p$ $\sim\Box\sim p \Leftrightarrow \Diamond p$ $\sim\Diamond\sim p \Leftrightarrow \Box p$

Analogy (Tpw: p is true at world w)
$\sim\forall wTpw \Leftrightarrow \exists w\sim Tpw$
$\sim\exists wTpw \Leftrightarrow \forall w\sim Tpw$
$\sim\forall w\sim Tpw \Leftrightarrow \exists wTpw$
$\sim\exists w\sim Tpw \Leftrightarrow \forall wTpw$

The analogy is based on the intuitive idea that necessity amounts to truth at *every* (accessible) possible world and possibility amounts to truth at *some* (accessible) possible world.

IV. TRANSLATIONS

p is sufficient for (entails) q (i.e., q is necessary for p)	$\Box(p \rightarrow q)$
p and q are compatible (compossible, consistent)	$\Diamond(p$ & $q)$
p is compatible (compossible, consistent) with q	$\Diamond(p$ & $q)$
p and q are incompatible (inconsistent, contraries)	$\sim\Diamond(p$ & $q)$ or $\Box(p \rightarrow \sim q)$

"If p is true, q must be true" is ambiguous between $\Box(p \rightarrow q)$ (necessity of the *consequence*) and $p \rightarrow \Box q$ (necessity of the *consequent*).

THE SYSTEM K

I. SEMANTICS

1. A *language* is a set of sentence letters (A, B, C,...). An *interpretation* of a language is an ordered triple $<W, R, v>$, where:
   1. W is a non-empty set (the set of all (*possible*) *worlds*);
   2. R is a binary relation on W (the relation of *accessibility* between worlds: Rww' stands for "w' is *accessible* from w");
   3. v is a function which assigns a truth value (either 'true' or 'false') to every sentence letter at every world (technically, to every sentence letter-possible world pair).

2. Given the truth values at a world of sentences p and q, the truth values at that world of $\sim p$, $(p \,\&\, q)$, $(p \vee q)$, $(p \rightarrow q)$, and $(p \leftrightarrow q)$ are determined by the standard truth tables for connectives.

3. At a given world w:
   1. $\Box p$ is true exactly if p is true at *every* world accessible from w;
   2. $\Diamond p$ is true exactly if p is true at *some* world accessible from w.

So, if *no* world is accessible from w, then, for every sentence p, $\Box p$ is true and $\Diamond p$ is false at w.

4. An argument is (*semantically*) *K-valid* exactly if, for every interpretation of its language, at every world of the interpretation at which all premises are true the conclusion is also true.

II. SEMANTIC TABLEAUX (TRUTH TREES)

Each line of the tableau is:
1. Either of the form p, i, where p is a sentence and i is natural number (such a line stands for the claim that the sentence p is true at world w_i);
2. Or of the form irj, where i and j are natural numbers (such a line stands for the claim that world w_j is accessible from world w_i).

Step 1: List vertically the premises and the *negation* of the conclusion, each followed by ', 0'.

Step 2: Check complex sentences and decompose them by using DN and:

(1) Indexed versions of classical decomposition rules. E.g.:

$p \,\&\, q, i$
p, i
q, i

$p \vee q, i$	
p, i	q, i

(2) Two new *whole-line* decomposition rules:

$\Box p, i$
irj
p, j

$\Diamond p, i$
irj
p, j
(j new to the branch)

- Do *not* check any *necessity* sentence that you decompose, but rather put '$*j$' next to it *every* time you decompose it (for different natural numbers j).
- The Modal Negation replacement rule may also be used.

Step 3: Stop if and only if:

1. Either every branch is *closed* (i.e., it contains, for some—maybe complex—sentence p and for some number i, both p, i and $\sim p$, i), in which case the argument is *valid*;
2. Or at least one *open* (i.e., not closed) branch is *finished* (i.e., all complex *non-necessity* sentences in the branch have been checked, and in the branch every possible application of the necessity decomposition rule has been made), in which case the argument is *invalid*.

PHILOSOPHY 512: PETER B. M. VRANAS
MODAL LOGIC HANDOUT 5

THE SYSTEM K: NATURAL DEDUCTION

I. MODAL VERSIONS OF THE CLASSICAL INFERENCE RULES

Take any classical inference rule (namely CE, CI, MP, MT, HS, DS, DA, or CD). Prefix its premise(s) and its conclusion by (maybe different) *modal prefixes*—namely finite non-empty strings of boxes and/or diamonds—*of the same length*. What results is a modal version of the classical inference rule exactly if two conditions hold:

(1) *Every* ***box*** *in the modal prefix of the conclusion is preserved in the modal prefix of* ***every*** *premise.*

(2) *Every* ***diamond*** *in the modal prefix of the conclusion is preserved in the modal prefix of* ***exactly one*** *premise.*

II. EXAMPLE: MODAL MODUS PONENS (MMP)

Argument	Instance of MMP?	Argument	Instance of MMP?
$\Box(A \to B)$ $\Box A$ $\Box B$	Yes	$\Diamond(A \to B)$ $\Diamond A$ $\Diamond B$	No: the diamond in the conclusion is preserved in *two* premises.
$\Box(A \to B)$ $\Diamond A$ $\Diamond B$	Yes	$\Box(A \to B)$ $\Box A$ $\Diamond B$	No: the diamond in the conclusion is preserved in *no* premise.
$\Diamond(A \to B)$ $\Box A$ $\Diamond B$	Yes	$\Box(A \to B)$ $\Diamond A$ $\Box B$	No: the box in the conclusion is not preserved in every premise.
$\Box\Box(A \to B)$ $\Box\Box A$ $\Box\Box B$	Yes	$\Box\Box(A \to B)$ $\Box\Box A$ $\Box B$	No: not all modal prefixes have the same length.
$\Box\Diamond\Box(A \to B)$ $\Diamond\Box\Box A$ $\Diamond\Diamond\Box B$	Yes	$\Box\Diamond\Box(A \to B)$ $\Diamond\Box\Box A$ $\Box\Diamond\Box B$	No: the first box in the conclusion is not preserved in every premise.

III. PREFIXED SUBPROOFS

1. A CP or RAA subproof may be prefixed by a string of n (≥ 1) boxes, abbreviated as '$\Box^n$'.

2. For a prefixed CP subproof, the CP assumption is discharged at a line of the form $\Box^n(p \to q)$, where p is the assumption [if the assumption is *empty* (a blank line), it is discharged at a line of the form $\Box^n q$] and q is the line immediately before the assumption is discharged.

3. For a prefixed RAA subproof, the RAA assumption is discharged at a line of the form $\Box^n{\sim}p$, if the assumption is p, or $\Box^n p$, if the assumption is ${\sim}p$; the line immediately before the assumption is discharged must be of the form q & ${\sim}q$.

4. Inside a prefixed subproof, lines from outside the subproof may be used only by:

Reiteration (Reit): If $\Box^n p$ is immediately outside a subproof prefixed by $\Box^n$, then p can be written immediately inside the subproof.

EXTENSIONS OF K

I. CONSTRAINTS ON THE ACCESSIBILITY RELATION

1. Extendability (η): $\forall w \exists w' Rww'$
2. Reflexivity (ρ): $\forall w Rww$
3. Symmetry (σ): $\forall w \forall w'(Rww' \to Rw'w)$
4. Transitivity (τ): $\forall w \forall w' \forall w''((Rww' \,\&\, Rw'w'') \to Rww'')$
5. Euclideanness (ε): $\forall w \forall w' \forall w''((Rww' \,\&\, Rww'') \to Rw'w'')$
6. Universality (υ): $\forall w \forall w' Rww'$

ρ entails η. στ entails ε. ρστ is equivalent to ρε (and to ηστ).

II. SEMANTICS FOR EXTENSIONS OF K

1. Given a combination κ of constraints on the accessibility relation, a *κ-interpretation* of a language is an interpretation whose accessibility relation satisfies those constraints. For example, a στ-interpretation is an interpretation whose accessibility relation is symmetric and transitive.

2. An argument *Kκ-valid* exactly if, for every κ-interpretation of its language, at every world of the interpretation at which all premises are true the conclusion is also true.

3. Since every κ-interpretation is an interpretation but not vice versa, *every K-valid argument is Kκ-valid but not nice versa: adding constraints increases* (*more precisely*: *does not decrease*) *the number of valid arguments*. In this sense, the system Kκ is an *extension* of K. For example, Kρ is an extension of K. Similarly, Kρσ is an extension of Kρ, Kρστ is an extension of Kρσ, and so on.

4. Some extensions of K are known by standard names:

Extension of K	Kη	Kρ	Kρσ	Kρτ	Kρστ=Kρε=Kυ
Standard name	D	T (or M)	B	S4	S5

In S4, any string of consecutive boxes can be reduced to a single box, and any string of consecutive diamonds can be reduced to single diamond. In S5, any string of consecutive boxes and/or diamonds can be reduced to the last symbol in the string.

III. AXIOMATIC CHARACTERIZATION OF EXTENSIONS OF K

A *system* (of modal logic) is a set of sentences (of a given language) closed under classical consequence. The system Kκ is the set of all and only those sentences (of a given language) that are true at every world of every κ-interpretation of the language. Alternatively, each system in the table below can be characterized by (i.e., as the smallest normal system containing every instance of) the corresponding axiom schema. [A system is *normal* exactly if it contains MN and, whenever it contains $(p_1 \,\&\, \ldots \,\&\, p_n) \to p$, it also contains $(\Box p_1 \,\&\, \ldots \,\&\, \Box p_n) \to \Box p$ ($n \geq 0$).]

System	Accessibility relation	Axiom schema	Name	Equivalent schema
K	No constraint	$\Box(p \to q) \to (\Box p \to \Box q)$	K	—
Kη	Extendable (serial)	$\Box p \to \Diamond p$	D	—
Kρ	Reflexive	$\Box p \to p$	T	$p \to \Diamond p$
Kτ	Transitive	$\Box p \to \Box\Box p$	4	$\Diamond\Diamond p \to \Diamond p$
Kσ	Symmetric	$p \to \Box\Diamond p$	B	$\Diamond\Box p \to p$
Kε	Euclidean	$\Diamond p \to \Box\Diamond p$	5	$\Diamond\Box p \to \Box p$

Similarly, Kρτ can be characterized by both axiom schemata T and 4, and so on.

IV. SEMANTIC TABLEAUX (TRUTH TREES)

Additional rules (they generate more applications of the necessity decomposition rule):

Name	η	ρ	σ	τ	ε
Rule	(*i* already in the branch)	(*i* already in the branch)	*irj*	*irj* *jrk*	*irj* *irk*
	irj	*iri*	*jri*	*irk*	*jrk*

For η, *j* is any natural number *new* to the branch.

An open branch is finished only if in the branch every possible (non-redundant) application of the relevant additional rules has been made.

V. NATURAL DEDUCTION

1. Inference Rules:

Accessibility relation	Rule	Name	Rule	Name
Extendable (serial)	$\frac{\Box p}{\Diamond p}$	η		
Reflexive	$\frac{\Box p}{p}$	ρ	$\frac{p}{\Diamond p}$	ρ′
Transitive	$\frac{\Box p}{\Box\Box p}$	τ	$\frac{\Diamond\Diamond p}{\Diamond p}$	τ′
Symmetric	$\frac{p}{\Box\Diamond p}$	σ	$\frac{\Diamond\Box p}{p}$	σ′
Euclidean	$\frac{\Diamond p}{\Box\Diamond p}$	ε	$\frac{\Diamond\Box p}{\Box p}$	ε′

2. Prefixed Versions of the Inference Rules:

Take any of the above inference rules and prefix both its premise and its conclusion by the *same* modal prefix α. What results is the α-version of the inference rule (named by prefixing the name of the rule by α).

Examples: From $\Box\Diamond\Box\Diamond A$ one can derive in one step $\Box\Diamond A$ by applying □σ′, and from $\Diamond\Box\Diamond\Diamond A$ one can derive in one step $\Diamond\Box\Diamond A$ by applying ◇□τ′.

3. Versions of Reiteration:

τ-Reiteration (τ-Reit): If $\Box p$ is immediately outside a subproof prefixed by $\Box^n$, then $\Box p$ can be written immediately inside the subproof.

ε-Reiteration (ε-Reit): If $\Diamond p$ is immediately outside a subproof prefixed by $\Box^n$, then $\Diamond p$ can be written immediately inside the subproof.

σ-Reiteration (σ-Reit): If p is immediately outside a subproof prefixed by $\Box^n$, then $\Diamond^n p$ can be written immediately inside the subproof.

HOW TO CHOOSE A SYSTEM

I. THE QUESTION: WHICH SYSTEM OF MODAL LOGIC IS CORRECT?

1. An *L-instantiation* of a *formula* (i.e., a sentence) of a given language of propositional modal logic is the *proposition* expressed by the formula when (1) the sentence letters in the formula express propositions and (2) any boxes and diamonds in the formula are understood as the propositional operators of *logical* necessity and possibility respectively. E.g., an L-instantiation of the formula $\Box(A\ \&\ B)$ is the proposition that it is logically necessary that both Al wins and it is metaphysically possible that if Bob runs then Al wins. This example shows that the sentence letters need not express distinct "atomic" propositions; if they do, then call the L-instantiation *simple*. E.g., a simple L-instantiation of the formula $\Box(A\ \&\ B)$ is the proposition that it is logically necessary that both Al wins and Bob runs. One can similarly define (simple) *M-instantiations* (corresponding to *metaphysical* necessity and possibility), and so on.

2. A *law of logical necessity* is a formula with only true L-instantiations. (Similarly for *metaphysical* necessity etc.) A system S of modal logic is the *correct* system of logical necessity exactly if it coincides with the set of laws of logical necessity; i.e., exactly if:

1. (*Soundness*:) Every S-valid formula is a law of logical necessity;
2. (*Completeness*:) Every law of logical necessity is S-valid.

II. LOGICAL NECESSITY

1. A proposition P is *logically necessary* exactly if *every proposition having the same logical form as P is true* (informally, P is "true by logical form alone"). Two propositions have the same logical form if they are simple L- (or M-, etc.) instantiations of the same formula. E.g., the propositions that (P_1) it is logically possible that the sky is yellow and that (P_2) it is logically possible that the sky is green are simple L-instantiations of the formula $\Diamond A$ and thus have the same logical form. P_2 is true (similarly for P_1): some true proposition (e.g., the proposition that the sky is blue) has the same logical form as the proposition that the sky is green.

2. Theorem: ***The correct system of logical necessity is S5.***[1]

3. Corollary of soundness: *every L-instantiation of an S5-valid formula is logically necessary.* Indeed: if F is an S5-valid formula and P is an L-instantiation of F, then $\Box F$ is also S5-valid and the proposition that P is logically necessary is an L-instantiation of $\Box F$ and thus is true. The converse of the corollary fails: *some logically necessary propositions are not L-instantiations of any S5-valid formula.* An example is the (true, and thus, by soundness) logically necessary proposition that it is logically possible that the sky is green ($\Diamond A$ is not S5-valid).

[1] To prove soundness—i.e., to prove that *every S5-valid formula has only true L-instantiations*—it is enough to prove that every formula which is an instance of axiom schema K, T, or 5 has only true L-instantiations. To prove this for 5 (the cases of K and T are similar), it is enough to prove that, for every proposition P, if the proposition that P is logically necessary is logically possible, then P is logically necessary. So take any proposition P and suppose that the proposition that P is logically necessary is logically possible. Then some true proposition Q has the same logical form as the proposition that P is logically necessary. Then there is a proposition R having the same logical form as P such that Q is the proposition that R is logically necessary. Then any proposition T having the same logical form as P has the same logical form as R (since R has the same logical form as P) and is thus true (since the proposition Q that R is logically necessary is true). So P is logically necessary.

4. To the distinction between L-instantiations of S5-valid formulas and logically necessary propositions corresponds *a distinction between* *two kinds of logical entailment* of a proposition *Q* from a proposition *P*, depending on whether the proposition that if *P* is true then *Q* is true is an L-instantiation of an S5-valid formula (*narrow* logical entailment) or is logically necessary (*wide* logical entailment). E.g., the proposition *P* that it is logically impossible that the sky is green logically entails the proposition that the sky is green in the wide sense (because *P* is false and thus logically impossible) but not in the narrow sense.

III. CONCEPTUAL NECESSITY

1. A proposition is *conceptually necessary* (or *analytic*) exactly if *it can be expressed by a sentence of a natural language which is true in virtue of the meanings of the words in it*. Arguably, the proposition that if I am taller than you then you are not taller than me is conceptually but not logically necessary, and so is the proposition that my shirt is not green if it is yellow.

2. Arguably, the proposition expressed by the sentence "if I am a bachelor, then I am male" is conceptually necessary. *If* that proposition can also be expressed by the sentence "if I am unmarried and I am male, then I am male", then it is an L-instantiation of the S5-valid formula $(U \,\&\, M) \rightarrow M$ and is thus also logically necessary.

IV. METAPHYSICAL NECESSITY

1. A proposition is *metaphysically necessary* exactly if (it is true and) *it would have been true no matter how things might have been*. Arguably, the proposition that if I am a human being then I am not a bank account is metaphysically but neither logically nor conceptually necessary.

2. To the distinctions between logical, conceptual, and metaphysical necessity correspond *distinctions between* *logical, conceptual, and metaphysical entailment*: a proposition *P* logically/conceptually/metaphysically entails a proposition *Q* exactly if it is logically/conceptually/metaphysically necessary that *Q* is true if *P* is true. E.g., arguably the proposition that I am a human being metaphysically but neither logically nor conceptually entails the proposition that I am not a bank account.

V. PHYSICAL AND NOMOLOGICAL NECESSITY

1. A proposition is *physically (nomologically) necessary* exactly if *it is entailed by the laws of physics* (*of nature*). Since there are several kinds of entailment, there are several kinds of physical necessity. E.g., if it is a law of physics that only photons travel at the speed of light and it is metaphysically necessary that no photon is a composite object, then the proposition that no composite object travels at the speed of light is metaphysically but presumably not logically entailed by the laws of physics, and is then physically necessary in one sense but not in another.

2. The above understanding of physical necessity does not *reduce* physical to e.g. logical necessity: it has the consequence that the laws of physics are physically necessary, but their physical necessity can hardly consist in the trivial fact that they are entailed by themselves.

REVIEW OF CLASSICAL QUANTIFIED LOGIC

I. SYNTAX

1. Logical symbols: *connectives* (~, &, ∨, → , ↔), *quantifiers* (∀, ∃), *variables* (*x*, *y*, *z*,…), and *punctuation symbols* (left and right parentheses).

2. Non-logical symbols: *constants* (*a*, *b*, *c*,…) and *predicates* (*P*, *Q*, *R*,…). Each predicate has a number of *places*. A *language* is a set of non-logical symbols.

3. A *term* is a variable or a constant. An *atomic formula* consists of an *n*-place predicate followed by *n* terms. A *formula* is either an atomic formula or anything built up in finitely many steps from atomic formulas as follows: if *F* and *G* are formulas, then ~*F*, (*F* & *G*), (*F* ∨ *G*), (*F* → *G*), (*F* ↔ *G*), ∀*xF*, and ∃*xF* (for any variable *x*) are formulas. (Outermost parentheses may be omitted.)

4. A *subformula* of a given formula is any string of consecutive symbols within the given formula which is a formula. An *occurrence* of a variable *x* in a formula is *bound* exactly if it is part of a subformula beginning with '∀*x*' or '∃*x*' (otherwise, the occurrence of the variable in the formula is *free*). A *sentence* is a formula in which every occurrence of every variable is bound.

II. SEMANTICS

1. An *interpretation* of a language is an ordered pair whose first member is a non-empty set (the *domain* of quantification) and whose second member is a function which (1) assigns to every *constant* a member of the domain (the *denotation* of the constant) and (2) assigns to every *n*-place *predicate* a set of ordered *n*-tuples of members of the domain (the *extension* of the predicate).

2. An interpretation of a language assigns truth values to the sentences of the language as follows: (1) an atomic sentence is true (on the interpretation) exactly if the ordered *n*-tuple consisting of the denotations of the *n* constants following the predicate is in the extension of the predicate; (2) non-atomic sentences formed by using connectives are true or false as per the standard truth tables for connectives; and (3) a quantified sentence ∀*xF* [respectively: ∃*xF*] is true (on the interpretation) exactly if, for every [respectively: for some] member *d* of the domain, the sentence obtained by replacing every free occurrence of *x* in *F* with a constant denoting *d* is true (on the interpretation).

III. TRANSLATIONS

All *A* are *B*	∀*x*(*Ax* → *Bx*)
All *A* who are *C* are *B*	∀*x*((*Ax* & *Cx*) → *Bx*)
Only *A* are *B*	∀*x*(*Bx* → *Ax*)
Only *A* who are *C* are *B*	∀*x*(*Bx* → (*Ax* & *Cx*))
All and only *A* are *B*	∀*x*(*Ax* ↔ *Bx*)
All and only *A* who are *C* are *B*	∀*x*((*Ax* & *Cx*) ↔ *Bx*)
No *A* are *B*	∀*x*(*Ax* → ~*Bx*)
No *A* who are *C* are *B*	∀*x*((*Ax* & *Cx*) → ~*Bx*)
Some *A* are B	∃*x*(*Ax* & *Bx*)
Some *A* who are *C* are *B*	∃*x*((*Ax* & *Cx*) & *Bx*)

IV. REPLACEMENT RULE

Name	Abbrev.	Rule
Quantifier Negation	QN	$\sim\forall xF \Leftrightarrow \exists x\sim F$ $\sim\exists xF \Leftrightarrow \forall x\sim F$ $\sim\forall x\sim F \Leftrightarrow \exists xF$ $\sim\exists x\sim F \Leftrightarrow \forall xF$

V. WHOLE-LINE INFERENCE RULES

1. The *instance* of $\forall xF$ and of $\exists xF$ with respect to the constant *a* is the formula obtained from *F* by replacing all free occurrences of *x* in *F* with *a*.

2. A *new* constant is a constant that does not appear in any previous line of a proof or in the conclusion.

Name	Abbrev.	Rule
Universal Instantiation	UI	From a universal sentence one can derive any instance of it.
Existential Generalization	EG	An existential sentence can be derived from any instance of it.
Existential Instantiation	EI	From an existential sentence one can derive any instance of it with respect to a *new* constant.
Universal Generalization	UG	A universal sentence can be derived from a *subproof* that starts by introducing a *new* constant and ends with the instance of the universal sentence with respect to that constant. <u>Restriction</u>: Any constant that occurs in the universal sentence must not be introduced in the subproof.

VI. SEMANTIC TABLEAUX (TRUTH TREES)

1. The decomposition rules from classical propositional logic, DN, and the Quantifier Negation replacement rule may be used, as well as the following *whole-line* instantiation rules ($F_x(a)$ is the result of replacing all free occurrences of the variable *x* in *F* with the constant *a*):

$\forall xF$
$F_x(a)$ (*a* is *any* constant)

$\exists xF$
$F_x(b)$ (*b* is any constant *new* to the branch)

$\exists xF$			
$F_x(a_1)$	…	$F_x(a_n)$	$F_x(b)$
($a_1,\ldots, a_n$ are all the constants already in the branch, and *b* is any constant new to the branch)			

Do *not* check any *universal* sentence that you instantiate, but rather put '**a*' next to it, where *a* is the constant with respect to which you instantiate. Do this every time you instantiate the universal sentence (with respect to different constants).

2. Steps 1-3 of the method are essentially the same as in classical propositional logic, except that now an open branch is *finished* exactly if (1) all complex *non-universal* sentences in the branch have been checked, and (2) all *universal* sentences in the branch have been (a) instantiated *at least once*, and (b) instantiated with respect to *every* constant in the branch.

3. Extra rules of thumb:
   1. Instantiate existential sentences before universal ones.
   2. If possible, instantiate universal sentences with respect to existing (not new) constants.
   3. Use the branching existential instantiation rule only to avoid infinite trees.

TWO KINDS OF QUANTIFIERS

I. MOTIVATION

It seems that sometimes we quantify not only over existing objects: in "some famous Greek philosophers are dead or have not yet been conceived" we quantify over objects that do not currently exist, in "some fictional detective is smarter than every actual detective" we quantify over fictional in addition to actual objects, and in "no possible object is a round square" we quantify over all possible objects.

II. NOTATION AND TERMINOLOGY

Use '$\forall$' and '$\exists$' for the *non-restricted* ('outer', 'possibilist') universal and existential (or better: 'particular') quantifiers respectively, which range over *all* objects, and use '$\forall^e$' and '$\exists^e$' for the *restricted* ('inner', 'actualist') universal and existential quantifiers respectively, which range over all (currently) *existing* objects. (The choice of symbols is motivated by the fact that non-restricted quantifiers behave like classical quantifiers but for restricted quantifiers new rules are needed. The choice of the term 'non-restricted'—rather than 'unrestricted'—is motivated by the fact that totally unrestricted quantification is arguably paradoxical.)

III. TRANSLATIONS

$\mathcal{E}x$ stands for "x exists". (If *identity* is introduced, $\mathcal{E}x$ can be replaced with $\exists^e y(y = x)$.)

English	Translation	Equivalent Translation
All *A* are *B*	$\forall x(Ax \to Bx)$	—
All existing *A* are *B*	$\forall^e x(Ax \to Bx)$	$\forall x(\mathcal{E}x \to (Ax \to Bx))$
Some *A* are *B*	$\exists x(Ax \& Bx)$	—
Some existing *A* are *B*	$\exists^e x(Ax \& Bx)$	$\exists x(\mathcal{E}x \& (Ax \& Bx))$
Some fictional detective is smarter than every actual detective	$\exists x((Fx \& Dx) \& \forall^e y(Dy \to Sxy))$	$\exists x((Fx \& Dx) \& \forall y(\mathcal{E}y \to (Dy \to Sxy)))$

IV. TWO OBJECTIONS

Objection 1: Non-restricted quantifiers make no sense because merely possible objects do not exist and thus one cannot quantify over them.

Reply: Quantifying over merely possible objects does not commit one to the claim that such objects exist. "Some detective does not exist" should not be understood as "there exists a detective who does not exist".

Objection 2: The distinction between non-restricted and restricted quantifiers is a distinction without a difference, because everything (speaking non-restrictedly) exists, so non-restricted and restricted quantifiers range over the same objects.

Reply: The metaphysical thesis that everything exists is controversial. Even if the thesis turns out to be true, and thus the distinction between non-restricted and restricted quantifiers turns out not to make a difference, the distinction is needed because logic should not prejudge controversial metaphysical issues.

FREE LOGIC

I. SEMANTICS FOR RESTRICTED QUANTIFIERS

1. An *interpretation* of a language is now an ordered *triple*: its new member is a (maybe empty) subset E of the domain. (The domain is the set of *all* objects; E is the set of all *existing* objects.) The extension of the *existence* predicate $\mathcal{E}$ is always E, but the extensions of other predicates and the denotations of constants are *not* restricted to existing objects. (Hence the name '*free* logic': logic free of existential assumptions. It is controversial whether having a property requires existing, but the semantics does not assume that it does, because logic should not prejudge controversial metaphysical issues. Adding a 'negativity constraint' to the effect that only existing objects have properties results in *negative* free logic.)

2. A quantified sentence $\forall^e xF$ [respectively: $\exists^e xF$] is true on an interpretation exactly if, for every [respectively: for some] member d of E, the sentence obtained by replacing every free occurrence of x in F with a constant denoting d is true on the interpretation.

II. REPLACEMENT RULE FOR RESTRICTED QUANTIFIERS

Name	Abbrev.	Rule
Free Quantifier Negation	FQN	$\sim\forall^e xF \Leftrightarrow \exists^e x{\sim}F$ $\sim\exists^e xF \Leftrightarrow \forall^e x{\sim}F$ $\sim\forall^e x{\sim}F \Leftrightarrow \exists^e xF$ $\sim\exists^e x{\sim}F \Leftrightarrow \forall^e xF$

III. INFERENCE RULES FOR RESTRICTED QUANTIFIERS

Name	Abbr.	Rule	Restrictions
Free Universal Instantiation	FUI	$\forall^e xF$ ——— $\mathcal{E}a \rightarrow F_x(a)$	No restriction: a is *any* constant.
Free Existential Generalization	FEG	$\mathcal{E}a \;\&\; F_x(a)$ ——— $\exists^e xF$	No restriction: a is *any* constant.
Free Existential Instantiation	FEI	$\exists^e xF$ ——— $\mathcal{E}b \;\&\; F_x(b)$	b is a *new* constant.
Free Universal Generalization	FUG	Subproof that starts by introducing a *new* constant b and ends with $\mathcal{E}b \rightarrow F_x(b)$ ——— $\forall^e xF$	Any constant that occurs in F must not be introduced in the subproof.

$F_x(a)$ is the result of replacing all free occurrences of the variable x in F with the constant a.

IV. SEMANTIC TABLEAUX FOR RESTRICTED QUANTIFIERS

The instantiation rules from classical quantified logic are replaced with the following rules:

$\forall^e xF$	
$\sim\mathcal{E}a$	$F_x(a)$
(a is *any* constant)	

$\exists^e xF$
$\mathcal{E}b$ $F_x(b)$
(b is any constant *new* to the branch)

$\exists^e xF$			
$\mathcal{E}a_1$ $F_x(a_1)$	…	$\mathcal{E}a_n$ $F_x(a_n)$	$\mathcal{E}b$ $F_x(b)$
($a_1,\ldots, a_n$ are all the constants already in the branch, and b is any constant new to the branch)			

QUANTIFIED MODAL LOGIC

I. MOTIVATION

Some obviously valid arguments cannot be proven to be valid in propositional modal logic or in classical quantified logic. For example: "It is possible that the number zero is immaterial. Therefore, it is possible that something is immaterial."

II. SEMANTICS

1. An *interpretation* of a language is an ordered quadruple $<W, R, D, v>$, where (1) W is a non-empty set (the set of all *worlds*), (2) R is a binary relation on W (the *accessibility* relation), (3) D is a non-empty set, the *domain of the interpretation* (the set of all *objects*), and (4) v is a function which (i) assigns to every *constant* a member of D (the *denotation* of the constant), (ii) assigns to every n-place *predicate* at every world a set of ordered n-tuples of members of D (the *extension* of the predicate *at* the world), and (iii) assigns to every member w of W a (maybe empty) subset D_w of D, the *domain of the world* w (the set of all objects that *exist at* w).

2. The union of the domains of all worlds need not exhaust the domain of the interpretation: some (impossible) objects may not exist at any world. On the other hand, the domains of all worlds may be identical to the domain of the interpretation (*constant-domain* interpretation). The extension of the *existence* predicate $\mathcal{E}$ at any world is identical to the domain of the world, but the extensions of other predicates at a world are *not* restricted to objects that exist at the world.

3. An atomic sentence is true (on an interpretation) at a given world exactly if the ordered n-tuple consisting of the denotations of the n constants following the predicate is in the extension of the predicate at the world. Non-atomic sentences formed by using connectives or modal operators are true or false (on the interpretation) at a world as per the standard rules.

4. On a given interpretation, for a given world w:

1. $\forall xF$ [respectively: $\exists xF$] is true at w exactly if, for every [respectively: for some] member of the *domain of the interpretation*, the sentence obtained by replacing every free occurrence of x in F with a constant denoting d is true at w;
2. $\forall^e xF$ [respectively: $\exists^e xF$] is true at w exactly if, for every [respectively: for some] member of the *domain of* w, the sentence obtained by replacing every free occurrence of x in F with a constant denoting d is true at w.

III. TRANSLATIONS

English sentences involving both quantifiers and modal expressions are often highly ambiguous.

English	Possible translations
It is impossible for a number to be located in space	$\Box\forall x(Nx \to {\sim}Lx)$, $\forall x\Box(Nx \to {\sim}Lx)$, $\forall x(Nx \to \Box{\sim}Lx)$
Everything is necessarily material	$\Box\forall^e xMx$, $\forall^e x\Box Mx$, $\forall x\Box Mx$
Every mental process is possibly caused by a brain process	$\forall^e x(Mx \to \Diamond\exists^e y(By \,\&\, Cyx))$, $\forall^e x(Mx \to \exists^e y(By \,\&\, \Diamond Cyx))$

IV. THE BARCAN AND BURIDAN FORMULAS

Valid?	Non-restricted versions		Restricted Versions	Valid exactly if:
YES	$\forall x\Box F \to \Box\forall xF$ $\Diamond\exists xF \to \exists x\Diamond F$	Barcan Formulas	$\forall^e x\Box F \to \Box\forall^e xF$ $\Diamond\exists^e xF \to \exists^e x\Diamond F$	$Rww' \to D_{w'} \subseteq D_w$ (*contracting* domains). I.e.: $\forall x(\sim\mathcal{E}x \to \Box\sim\mathcal{E}x)$.
	$\Box\forall xF \to \forall x\Box F$ $\exists x\Diamond F \to \Diamond\exists xF$	Converse Barcan formulas	$\Box\forall^e xF \to \forall^e x\Box F$ $\exists^e x\Diamond F \to \Diamond\exists^e xF$	$Rww' \to D_w \subseteq D_{w'}$ (*expanding* domains). I.e.: $\forall x(\mathcal{E}x \to \Box\mathcal{E}x)$.
	$\Diamond\forall xF \to \forall x\Diamond F$ $\exists x\Box F \to \Box\exists xF$	Buridan formulas	$\Diamond\forall^e xF \to \forall^e x\Diamond F$ $\exists^e x\Box F \to \Box\exists^e xF$	
NO	$\forall x\Diamond F \to \Diamond\forall xF$ $\Box\exists xF \to \exists x\Box F$	Converse Buridan formulas	$\forall^e x\Diamond F \to \Diamond\forall^e xF$ $\Box\exists^e xF \to \exists^e x\Box F$	

- To see intuitively which *non*-restricted versions of the above formulas are valid, recall that $\Box\forall xPx$ is analogous to $\forall w\forall xPxw$ (every object has the property P at every world) etc. (Then isn't the box redundant? How does $\Box\forall xPx$ differ from $\forall xPx$? The latter is indeed about all possible objects, but says only that they have the property P at the *actual* world.) So $\forall x\Diamond Px \to \Diamond\forall xPx$ is invalid (i.e., not true at every world of every interpretation) "because" $\forall x\exists wPxw \to \exists w\forall xPxw$ is invalid in classical quantified logic.
- To see intuitively why the restricted versions of the Barcan formulas are valid under the contracting domains condition, consider the temporal analogy: if all currently existing objects will always have property P ($\forall^e x\Box Px$) and no new objects will ever come into existence ($\forall x(\sim\mathcal{E}x \to \Box\sim\mathcal{E}x)$), then it will always be the case that all (then) existing objects have property P ($\Box\forall^e xPx$). Similarly for the restricted versions of the Converse Barcan formulas and the expanding domains condition.
- To see that the expanding domains condition follows from the restricted versions of the Converse Barcan formulas, replace F with $\mathcal{E}x$ to get: $\Box\forall^e x\mathcal{E}x \to \forall^e x\Box\mathcal{E}x$. The antecedent is true, and the consequent is equivalent to the expanding domains condition.

V. REMARK

If a sentence or an argument in which some restricted quantifiers occur and the existence predicate does not occur is valid, then the corresponding sentence or argument in which all restricted quantifiers are replaced with the corresponding non-restricted ones is also valid.

ACTUALISM AND POSSIBILISM

I. INTRODUCTION

I could have failed to exist. So there is a world at which I do not exist, at which I am a *non-existent object*. Moreover, I could have had a son. So there is a world at which I have a son. If this (possible) son of mine is distinct from every object that actually exists, then he is a *non-actual object*. The actualism/possibilism debates concern the possibility, the existence, and the ontological status of non-existent and non-actual objects. (i) According to *weak possibilism*, non-existent or non-actual objects are *possible* (I *could* have failed to exist; I *could* have had a son distinct from every object that actually exists); *domain-inclusion actualism* denies this. (ii) According to *strong possibilism*, non-existent or non-actual objects *exist simpliciter*—or have *being*, an ontological status 'lesser' than existence. Since by definition non-existent and non-actual objects do not (actually) exist, strong possibilism relies on a distinction between existence simpliciter—or being—and (actual) existence, a distinction that actualism denies.

II. WEAK POSSIBILISM & DOMAIN-INCLUSION ACTUALISM

1. Weak possibilism comes in two main versions. Both versions can be formulated by "possibly, something does not actually exist".

- If the word 'actually' is used *non-rigidly* (so that it is redundant, and at every world 'the actual word' refers to *that* world), we get non-rigid weak possibilism: (1) $\Diamond\exists x{\sim}\mathcal{E}x$ ("non-existent objects are possible").
- If the word 'actually' is used *rigidly* (as I normally use it, so that at every world 'the actual world' refers to *our* world), we get rigid weak possibilism: (2) $\Diamond\exists x{\sim}@\mathcal{E}x$ ("non-actual objects are possible").

(1) does not entail (2), but (2) entails (1) if the accessibility relation is universal.

2. Non-rigid weak possibilism should be hardly controversial: $\Diamond\exists x{\sim}\mathcal{E}x$ follows (by Modal Existential Generalization) from (1*) $\Diamond{\sim}\mathcal{E}s$ ("possibly, it is not the case that Socrates exists"). [This is the gist of the 'Classical Argument'; Plantinga apparently misses this.]

3. Rigid weak possibilism is plausible: $\Diamond\exists x{\sim}@\mathcal{E}x$ follows from (2*) $\Diamond\exists^e x{\sim}@\mathcal{E}x$ ("non-actual objects could have existed"), which in turn follows from two premises: (i) I could have had a son ($\Diamond\exists^e xSx$), and (ii) no object that actually exists could have been a son of mine ($\forall x(@\mathcal{E}x \rightarrow \Box{\sim}Sx)$) [because any son of mine would have to originate from my sperm ($\forall x\Box(Sx \rightarrow Ox)$), but no object that actually exists originates from my sperm ($\forall x(@\mathcal{E}x \rightarrow {\sim}Ox)$), and nothing that does not originate from my sperm could have originated from my sperm ($\forall x({\sim}Ox \rightarrow \Box{\sim}Ox)$)].

4. Rigid weak possibilism entails—and is thus equivalent to—(2*) if *everything could have existed* ($\forall x\Diamond\mathcal{E}x$); i.e., if the domain of the interpretation is the union of the domains of all accessible worlds.

5. Rigid weak possibilism entails (and, if the accessibility relation is reflexive, is entailed by): (2′) $\exists x{\sim}@\mathcal{E}x$ ("something does not actually exist"). (2′) is *materially*—but not *logically*—equivalent to: (1′) $\exists x{\sim}\mathcal{E}x$ ("something does not exist"). So (1′) is true if rigid weak possibilism is true. (1′) should be distinguished from $\exists^e x{\sim}\mathcal{E}x$ ("some existing object does not exist"), which is trivially false. [This corresponds to Bennett's point that one needs *non*-restricted quantifiers in order to formulate (some versions of) actualism and possibilism.]

6. Negating each possibilist claim yields an actualist claim (thick arrows indicate entailment):

Possibilist claims			Actualist claims		
Non-rigid		**Rigid**	**Non-rigid**		**Rigid**
(1*) $\Diamond{\sim}\mathcal{E}s$ (Possibly, Socrates does not exist)		(2*) $\Diamond\exists^e x{\sim}@\mathcal{E}x$ (Non-actual objects could have existed)	(1_A*) $\Box\mathcal{E}s$ (Necessarily, Socrates exists)		(2_A*) $\Box\forall^e x@\mathcal{E}x$ (Necessarily, every existing object actually exists)
$\Downarrow$		$\Downarrow$ ($\Uparrow$ given $\forall x\Diamond\mathcal{E}x$)	$\Uparrow$		$\Uparrow$ ($\Downarrow$ given $\forall x\Diamond\mathcal{E}x$)
(1) $\Diamond\exists x{\sim}\mathcal{E}x$ (Non-existent objects are possible)	($\Leftarrow$ given υ)	(2) $\Diamond\exists x{\sim}@\mathcal{E}x$ (Non-actual objects are possible)	(1_A) $\Box\forall x\mathcal{E}x$ (Necessarily, everything exists)	($\Rightarrow$ given υ)	(2_A) $\Box\forall x@\mathcal{E}x$ (Necessarily, everything actually exists)
($\Uparrow$ given ρ)		$\Downarrow$ ($\Uparrow$ given ρ)	($\Downarrow$ given ρ)		$\Uparrow$ ($\Downarrow$ given ρ)
(1′) $\exists x{\sim}\mathcal{E}x$ (Something does not exist)	$\leftrightarrow$	(2′) $\exists x{\sim}@\mathcal{E}x$ (Something does not actually exist)	(1_A′) $\forall x\mathcal{E}x$ (Everything exists)	$\leftrightarrow$	(2_A′) $\forall x@\mathcal{E}x$ (Everything actually exists)

7. (2_A′)—and, given reflexivity, (2_A)—amounts to the claim that the domain of the interpretation is *included* in the domain of the actual world, from which the claim—which amounts to (2_A*)—follows that the domain of every (accessible) world is included in the domain of the actual world. So (2_A*), (2_A), and (2_A′) can be called *domain-inclusion* actualist claims. [The contrast between (1_A) and (2_A) corresponds to Bennett's contrast between *w*-ism and @-ism.]

8. (1_A) should be rejected: it entails (1_A*). If (2_A*) is false, then (2_A), (2_A′), and (1_A′) are false. So domain-inclusion actualists must reject (2*). *Proxy actualists* reject (2*) by claiming, e.g., that although I could *not* have had a son distinct from every object that actually exists, some actually existing abstract object could have been a son of mine (and thus could have been concrete).

III. STRONG POSSIBILISM & NON-DOMAIN-INCLUSION ACTUALISM

1. Strong possibilism comes in two kinds. Both can be formulated by the slogan "there are non-actual objects" (or "there are non-existent objects", in the non-rigid versions).

- *Classical possibilists* understand the slogan as "non-actual objects have *being*".
- *Lewisian possibilists* understand the slogan as "non-actual objects *exist simpliciter*". On this view, actual and non-actual objects have the same ontological status: both exist simpliciter. But actual objects exist *at* (i.e., are parts of) the actual world, while non-actual objects exist at other worlds. Lewis understands worlds as *universes*, objects like our universe.

2. Non-domain-inclusion actualism is the thesis that *non-actual objects do not exist simpliciter*. Actualists typically (i) understand existence simpliciter as actual existence (something exists simpliciter exactly if it is part of our universe) and (ii) understand worlds as (actually existing) *abstract representational devices*—and thus distinguish our universe (which is what Lewis calls 'the actual world') from the actual world (namely the device that represents our universe). To exist at a world is to be represented by the world as existing, not to be part of a universe.

3. One can introduce a predicate $\mathcal{E}$* for existence simpliciter and [*pace* Bennett] formulate Lewisian possibilism as $\exists x(\mathcal{E}^*x \;\&\; {\sim}@\mathcal{E}x)$ and non-domain-inclusion actualism as $\forall x({\sim}@\mathcal{E}x \rightarrow {\sim}\mathcal{E}^*x)$. The extension of $\mathcal{E}$* at every world is the domain of the interpretation according to Lewisian possibilists, but is the domain of the actual world according to some actualists.

4. One might argue that the notion of existence simpliciter is confused. Compare: once one realizes that propositions in general have different truth values at different worlds, one gives up the notion of truth simpliciter and replaces it with the notion of truth *at* a world. Similarly, once one realizes that objects in general exist at some worlds but not at others, arguably one should give up the notion of existence simpliciter and replace it with the notion of existence *at* a world. Of course one can *define* existence simpliciter as existence at the *actual* world or at *some* world, but then Lewisian possibilism comes out as trivially false or trivially true respectively.

PHILOSOPHY 512:
MODAL LOGIC

PETER B. M. VRANAS
HANDOUT 13

SEMANTIC TABLEAUX AND NATURAL DEDUCTION FOR QUANTIFIED MODAL LOGIC

I. SEMANTIC TABLEAUX FOR NON-RESTRICTED QUANTIFIERS

Use the semantic tableau rules for propositional modal logic, as well as QN and the indexed versions of the instantiation rules for classical quantified logic:

$\forall xF, i$
$F_x(a), i$
(a is *any* constant)

$\exists xF, i$
$F_x(b), i$
(b is any constant *new* to the branch)

$\exists xF, i$			
$F_x(a_1), i$	…	$F_x(a_n), i$	$F_x(b), i$
($a_1, \ldots, a_n$ are all the constants already in the branch, and b is any constant new to the branch)			

II. NATURAL DEDUCTION FOR NON-RESTRICTED QUANTIFIERS

1. Use the natural deduction rules for propositional modal and classical quantified (see 2 below) logic, as well as the following modal versions of the inference rules for classical quantified logic:

Name	Abbrev.	Rule
Modal Universal Instantiation	MUI	From a universal sentence *prefixed by any modal prefix* one can derive any instance of the universal sentence *prefixed by the same modal prefix*. E.g.: from $\Box\Diamond\Box\forall xPx$ one can derive $\Box\Diamond\Box Pa$.
Modal Existential Generalization	MEG	An existential sentence *prefixed by any modal prefix* can be derived from any instance of the existential sentence *prefixed by the same modal prefix*. E.g.: from $\Diamond\Box Pa$ one can derive $\Diamond\Box\exists xPx$.
Modal Existential Instantiation	MEI	From an existential sentence *prefixed by any string of diamonds* one can derive any instance of the existential sentence with respect to a *new* constant *prefixed by the same string of diamonds*. E.g.: from $\Diamond\Diamond\exists xPx$ one can derive $\Diamond\Diamond Pb$.
Modal Universal Generalization	MUG	A universal sentence *prefixed by any string of boxes* can be derived from a *subproof* that starts by introducing a *new* constant and ends with the instance of the universal sentence with respect to that constant *prefixed with the same string of boxes*. Restriction: Any constant that occurs in the universal sentence must not be introduced in the subproof.

2. Restriction on prefixed subproofs: any constant introduced (e.g., by FEI or MEI) in a *prefixed* subproof must not appear at the line where the assumption of the subproof is discharged.

III. SEMANTIC TABLEAUX FOR RESTRICTED QUANTIFIERS

Use the semantic tableau rules for propositional modal logic, as well as FQN and the indexed versions of the instantiation rules for *free* logic:

$\forall^e xF, i$	
$\sim\mathcal{E}a, i$	$F_x(a), i$
(a is *any* constant)	

$\exists^e xF, i$
$\mathcal{E}b, i$
$F_x(b), i$
(b is *new* to the branch)

$\exists^e xF, i$			
$\mathcal{E}a_1, i$ $F_x(a_1), i$	…	$\mathcal{E}a_n, i$ $F_x(a_n), i$	$\mathcal{E}b, i$ $F_x(b), i$
($a_1, \ldots, a_n$ are all the constants already in the branch, and b is any constant new to the branch)			

IV. NATURAL DEDUCTION FOR RESTRICTED QUANTIFIERS

Use the natural deduction rules for propositional modal logic and free logic (see II.2 above for a restriction), as well as the following modal versions of the inference rules for free logic:

Name	Abbrev.	Rule (π is any modal prefix)	Example
Modal Free Universal Instantiation	MFUI	$\dfrac{\pi\forall^{e}xF}{\pi(\mathcal{E}a \to F_x(a))}$ (a is *any* constant)	$\dfrac{\Diamond\Box\forall^{e}xPx}{\Diamond\Box(\mathcal{E}a \to Pa)}$
Modal Free Existential Generalization	MFEG	$\dfrac{\pi(\mathcal{E}a \ \&\ F_x(a))}{\pi\exists^{e}xF}$ (a is *any* constant)	$\dfrac{\Box\Diamond\Box(\mathcal{E}a \ \&\ Pa)}{\Box\Diamond\Box\exists^{e}xPx}$
Modal Free Existential Instantiation	MFEI	$\dfrac{\Diamond^{n}\exists^{e}xF}{\Diamond^{n}(\mathcal{E}b \ \&\ F_x(b))}$ (b is a *new* constant)	$\dfrac{\Diamond\Diamond\exists^{e}xPx}{\Diamond\Diamond(\mathcal{E}b \ \&\ Pb)}$
Modal Free Universal Generalization	MFUG	Subproof that starts by introducing a *new* constant b and ends with $\Box^{n}(\mathcal{E}b \to F_x(b))$ ——— $\Box^{n}\forall^{e}xF$ (any constant that occurs in F must not be introduced in the subproof)	Subproof of $\Box\Box(\mathcal{E}b \to Pb)$ ——— $\Box\Box\forall^{e}xPx$

THE ACTUALITY OPERATOR

I. MOTIVATION

How to translate "there is an accessible world at which everyone is poor who exists at that world but is rich at the actual world"? $\forall^{e}x(Rx \to \Diamond Px)$ and $\Diamond\forall^{e}x(Rx \to Px)$ will not do. Solution: introduce the actuality operator '@' ('actually') and translate the sentence as $\Diamond\forall^{e}x(@Rx \to Px)$.

II. SYNTAX AND SEMANTICS

1. For any formula F, $@F$ is a formula (informally read as "actually, F").

2. On a given interpretation, for a given world w, $@F$ is true at w exactly if F is true at the world designated as the actual world (specified by the interpretation).

III. SEMANTIC TABLEAUX

Take $w_{\#}$ to be the actual world. New decomposition rules:

$\dfrac{@F,\ i}{F,\ \#}$ $\qquad$ $\dfrac{\sim @F,\ i}{\sim F,\ \#}$

IV. REPLACEMENT RULES

Name	Abbr.	Rule
Actuality Repetition	AR	$@@F \Leftrightarrow @F$
Actuality Negation	AN	$\sim @F \Leftrightarrow @\sim F$
Actuality Conjunction	AC	$@(F \ \&\ G) \Leftrightarrow @F \ \&\ @G$

V. INFERENCE RULES

Name	Abbr.	Rule	
Necessity of Actuality	NA	$\dfrac{@F}{\Box^{n}@F}$	$\dfrac{\Diamond^{n}\sim @F}{\sim @F}$
Necessity of non-actuality	NN	$\dfrac{\sim @F}{\Box^{n}\sim @F}$	$\dfrac{\Diamond^{n}@F}{@F}$

EXISTENCE AND DESIGNATION

I. THE QUESTION

Is it the case that (Q) some terms refer to non-existent objects? (One might say no because there cannot be any non-existent objects, but this worry was addressed in Handout 12.)

II. ARGUMENTS FOR Q

1. Linguistic evidence: It seems that 'Pegasus' refers to Pegasus and that 'dragons' refers to dragons, although Pegasus and dragons do not exist.

2. Analogy: One can *worship*, *admire*, *postulate*, *fear* non-existent objects; why not *refer*?

III. OBJECTION 1

It is inconsistent to claim that (1) 'Pegasus' refers to Pegasus, (2) if 'Pegasus' refers to Pegasus, then Pegasus exists, and (3) Pegasus does not exist; so the evidence (1)&(3) in favor of Q is suspect, since adding to it the plausible claim (2) leads to a contradiction.

IV. REPLIES TO OBJECTION 1

1. Reply 1: (2) relies on the general claim that (2*) if something refers to Pegasus, then Pegasus exists. But (2*) is self-defeating, since it itself refers to Pegasus, so it leads to the absurd conclusion that Pegasus exists. Response: One can deny that (2*) refers to Pegasus; indeed, a proponent of (2*) who accepts (3) *must* deny that (2*) refers to Pegasus.

2. Reply 2: (2) just is the negation of (1)&(3), so of course adding (2) to (1)&(3) yields a contradiction. Affirming (2) is question-begging: it amounts to just denying the evidence (1)&(3), not to adducing an argument against it. Response: But similarly, affirming (1)&(3) amounts to just denying (2), so it is a stalemate.

V. OBJECTION 2

Sentences in which *apparent* reference to non-existent objects is made can be paraphrased so that the appearance vanishes. For example, "dragons do not exist" ($\sim\exists^e xDx$) can be paraphrased as "no existing object has the property of being a dragon", and "Pegasus does not exist" (*apparently* $\sim\mathcal{E}p$) can be paraphrased as "no existing object is the winged horse of Greek mythology".

VI. REPLIES TO OBJECTION 2

Reply 1: It will not do to replace names with definite descriptions, given Kripke's well-known objections to the description theory of names. Reply 2: Intuitively, "Pegasus is a pig" is false but "Pegasus has wings" is true; but replacing 'Pegasus' with a definite description leads to the result that both sentences are false. Reply 3: There is no problem with translating "Socrates does not exist" as $\sim\exists^e x(x = s)$, so those who advocate *not* translating "Pegasus does not exist" as $\sim\exists^e x(x = p)$ face a dilemma: either they must say that how we translate sentences depends on whether the sentences *are true* (unpalatable: we want to be able to translate "God does not exist" without knowing whether God exists), or they must give up constants in translations altogether (e.g., they must translate "Socrates does not exist" without using any constant).

VII. SALMON'S ARGUMENT FOR Q

Let 'Noman' refer to the individual that would have existed if the sperm from which I originate had fertilized the ovum from which you originate. 'Noman' refers to a non-existent object.

IDENTITY

I. SYNTAX AND SEMANTICS

1. New logical symbol: ‘=’, the *identity sign*. It is a two-place predicate, and thus it is used like every other predicate to form (atomic and non-atomic) formulas, but by convention ‘$=tt'$’is written as ‘$(t = t')$’ and ‘$\sim(t = t')$’ may be abbreviated as ‘$(t \neq t')$’, where t and t' are any terms (i.e., variables or constants). (The parentheses may be omitted.)

2. On a given interpretation, an atomic sentence $a = b$ is true (at a given world) exactly if the constants a and b denote the same member of the domain of the interpretation.

II. TRANSLATIONS

There are at least two gods	$\exists^e x \exists^e y((Gx \,\&\, Gy) \,\&\, x \neq y)$
There are at least three gods	$\exists^e x \exists^e y \exists^e z(((Gx \,\&\, Gy) \,\&\, Gz) \,\&\, ((x \neq y \,\&\, y \neq z) \,\&\, z \neq x))$
There is at most one god	$\forall^e x \forall^e y((Gx \,\&\, Gy) \to x = y)$
There are at most two gods	$\forall^e x \forall^e y \forall^e z(((Gx \,\&\, Gy) \,\&\, Gz) \to ((x = y \vee y = z) \vee z = x))$
There is exactly one god	$\exists^e x(Gx \,\&\, \forall^e y(Gy \to y = x))$
There are exactly two gods	$\exists^e x \exists^e y(((Gx \,\&\, Gy) \,\&\, x \neq y) \,\&\, \forall^e z(Gz \to (z = x \vee z = y)))$
The only immortal god is Zeus	$(Gz \,\&\, \sim Mz) \,\&\, \forall^e x((Gx \,\&\, \sim Mx) \to x = z)$

III. SEMANTIC TABLEAUX FOR IDENTITY

Identity substitution
$F_x(a), i$ $a = b, i$ [or $b = a, i$]
$F_x(b), i$ (F is any *atomic* formula except $a = b$)

Identity invariance
$a = b, i$
$a = b, j$ (j already in the branch)

A branch is closed if it contains $a \neq a, i$ or both $a = b, i$ and $b \neq a, i$. An open branch is finished only if it includes all possible (non-redundant) applications of the two identity rules. F could be non-atomic, but this would make it harder to find finished open branches.

IV. REPLACEMENT RULE FOR IDENTITY

Name	Abbrev.	Rule	Remark
Identity Symmetry	I Sym	$t = t' \Leftrightarrow t' = t$	This rule is redundant but convenient.

V. INFERENCE RULES FOR IDENTITY

Name	Abbr.	Rule	Remarks
Identity Reflexivity	IR	$\dfrac{}{a = a}$	This rule can be used anywhere, even inside a prefixed subproof. (a is *any* constant.)
Identity Substitution	IS	$\dfrac{F_x(a) \quad a = b \text{ [or } b = a\text{]}}{F_x(b)}$	This rule says that *some or all* occurrences of a in F can be replaced with b, and *also has modal versions*. (F need not be atomic.)
Necessity of Identity	NI	$\dfrac{a = b}{\Box^n(a = b)}$ $\quad \dfrac{\Diamond^n(a \neq b)}{a \neq b}$	This rule is redundant (given IR and IS) but convenient.
Necessity of Distinctness	ND	$\dfrac{a \neq b}{\Box^n(a \neq b)}$ $\quad \dfrac{\Diamond^n(a = b)}{a = b}$	This rule is redundant (given IR and IS) *if* the accessibility relation is symmetric.

NON-RIGID DESIGNATORS AND DE RE MODALITY

I. FOUR KINDS OF TERMS

1. So far we have treated terms and predicates differently: an interpretation assigned to a predicate an extension *at a world*, but assigned to a constant a denotation not relative to a world. But some terms (referring expressions, designators) denote different objects at different worlds. E.g., 'the 44th U.S. President' denotes Obama at the actual world but McCain at some other world. Moreover, some terms (e.g., 'the King of the USA') denote nothing at some worlds.

2. To say that term *t* denotes object *o* at world *w* is *not* to say that people at *w* standardly use *t* to denote *o*; it is rather to say that *we* use *t* to denote *o* when we talk about *w*. E.g., suppose that at some world *w* Plato has my name but I am called 'Plato'. Since I just said that at *w* Plato has my name, I used 'Plato' to denote at *w* the object who is actually a Greek philosopher, not myself.

3. Denotation at a world does not require existence at the world: if I say "consider a world *w* at which Plato does not exist", at *w* 'Plato' denotes Plato although at *w* Plato does not exist.

4. Besides variables (x, y, z,…) and constants (a, b, c,…), introduce terms of a third kind: *descriptors* (α, β, γ,…). An interpretation now assigns to constants and to descriptors objects (members of the domain of the interpretation) *at worlds*: it assigns to a constant the same object at every world, but it may assign to a descriptor different objects at different worlds and no object at some (or even at all) worlds. The truth value of an atomic formula at a given world depends on what any descriptors in the formula denote at that world; if some descriptor in an atomic formula denotes nothing at a given world, then the formula is false at that world.

5. Say that (on a given interpretation) a constant or descriptor is (1) *rigid* exactly if it denotes the same object at every world at which it denotes something, is (2) *non-rigid* otherwise, is (3) *total* exactly if it denotes something at every world, and is (4) *partial* otherwise. All constants are rigid and total, but for descriptors there are four possibilities:

Terms	Total	Partial
Rigid	'The smallest natural number'; 'Plato'	'The zygote from which Plato originates'
Non-rigid	'The number of elementary particles'	'The 44th U.S. President'; 'Miss America'

II. DE DICTO AND DE RE MODALITY

1. Sentences with both modal operators and non-rigid terms can be ambiguous. "The 44th U.S. President could have been a Republican" can be understood as (1) "at some possible world the 44th U.S. President is a Republican" (e.g., because McCain wins the election), or as (2) "the person who actually is the 44th U.S. President at some possible world is a Republican" (e.g., because Obama could have had different political views). (If all terms are rigid and total, the two readings are logically equivalent, so the ambiguity does not matter.) If π stands for the 44th U.S. President, $\Diamond R\pi$ corresponds to the former (*de dicto*) reading; but how to formalize the latter (*de re*) reading, which says of a given object that it has the property of being possibly a Republican?

2. (1) Can one formalize the *de re* reading as $\Diamond Ro$, where o is a constant denoting Obama at every world? No: $\Diamond Ro$ is only materially (not logically) equivalent to "the 44th U.S. President has the property of being possibly a Republican" (at worlds at which no 44th U.S. President exists, the latter sentence is false, but $\Diamond Ro$ may be true). (2) One might use Russell's theory of descriptions to formalize the *de dicto/de re* ambiguity as $\Diamond\exists x((Px \,\&\, Rx) \,\&\, \forall y(Py \to y = x))$ vs $\exists x((Px \,\&\, \Diamond Rx) \,\&\, \forall y(Ry \to y = x))$. But what about non-rigid terms that are not definite descriptions? (3) Better way: use *predicate abstraction* to represent complex properties.

PREDICATE ABSTRACTION

I. THE GENERAL IDEA

Predicate abstraction is a way of forming complex predicates. E.g., rather than translating the sentence “John is tall and happy” as the conjunction of “John is tall” and “John is happy” (Tj & Hj), introduce the predicate ‘being tall-and-happy’ ($\lambda x(Tx$ & $Hx)$: the property of being an x such that x is both tall and happy), and translate the sentence as “John has the property of being tall-and-happy” ($\lambda x(Tx$ & $Hx)j$: John is an x such that x is both tall and happy). Similarly, to translate the *de re* reading of the sentence “the President is possibly a Republican”, introduce the predicate ‘being possibly a Republican’ ($\lambda x(\Diamond Rx)$: the property of being an x such that x is possibly a Republican) and translate the sentence as “the President has the property of being possibly a Republican” ($\lambda x(\Diamond Rx)\pi$: the President is an x such that x is possibly a Republican). By contrast, the *de dicto* reading of the sentence is translated as “it is possible that the President has the property of being a Republican” ($\Diamond\lambda x(Rx)\pi$; equivalently, $\Diamond R\pi$).

II. SYNTAX AND SEMANTICS

1. New logical symbol: ‘λ’, the *abstraction quantifier*. New way of forming formulas: if F is a formula, x is a variable, and t is a term (i.e., a variable, a constant, or a descriptor), then λxFt is a formula (called an *abstraction formula*, and informally read as “the object actually denoted by t has the property of being an x such that F”). Every occurrence of x which is free in F is bound in λxFt. [λxF is called a *predicate abstract* and is informally read as “the property of being an x such that F”.] To increase readability, enclose F in parentheses; e.g., $\lambda x(\Diamond Rx)c$.

2. On a given interpretation, for a given world w: (1) if t denotes nothing at w, then λxFt is false at w, and (2) if t denotes object o at w, then λxFt is true at w exactly if $F_x(a)$ is true at w, where a is a constant that denotes o at every world. (So, e.g., $\lambda x({\sim}\mathcal{E}x)t$ is false but ${\sim}\lambda x(\mathcal{E}x)t$ is true at worlds at which t does not denote.)

III. EXAMPLES

Dx: x is a Democrat; Rx: x is a Republican; Axy: x admires y; π: the U.S. President; ϕ: the French President; o: Obama; s: Sarkozy.

English	Possible translations
The U.S. President is a Democrat but could have been a Republican	(1) $D\pi$ & $\Diamond R\pi$ (i.e.: $\lambda x(Dx)\pi$ & $\Diamond\lambda x(Rx)\pi$) (2) $\lambda x(Dx$ & $\Diamond Rx)\pi$
The U.S. President might have admired someone	(1) $\Diamond\exists xA\pi x$ [i.e.: $\Diamond\exists x\lambda y(Ayx)\pi$, or also $\Diamond\lambda y\exists x(Ayx)\pi$] (2) $\exists x\Diamond A\pi x$ [i.e.: $\exists x\Diamond\lambda y(Ayx)\pi$] (3) $\lambda y(\exists x\Diamond Ayx)\pi)$ [i.e.: $\exists x\lambda y(\Diamond Ayx)\pi$] (4) $\lambda y(\Diamond\exists xAyx)\pi)$
The U.S. President might have admired the French President	(1) $\Diamond A\pi\phi$ [i.e : $\Diamond\lambda x(\lambda y(Axy)\phi)\pi$] (2) $\lambda x(\Diamond Ax\phi)\pi$ [i.e.: $\lambda x(\Diamond\lambda y(Axy)\phi)\pi$]; true iff $\Diamond Ao\phi$ (3) $\lambda y(\Diamond A\pi y)\phi$ [i.e.: $\lambda y(\Diamond\lambda x(Axy)\pi)\phi$]; true iff $\Diamond A\pi s$ (4) $\lambda x(\lambda y(\Diamond Axy)\phi)\pi$ [i.e.: $\lambda y(\lambda x(\Diamond Axy)\pi)\phi$]; true iff $\Diamond Aos$

The *semantic* notions of a descriptor γ (1) denoting and (2) denoting *locally* rigidly (i.e., denoting the same object at each *accessible* world at which γ denotes) can be expressed *syntactically* by (1) $\lambda x(\mathcal{E}x)\gamma \vee \lambda x({\sim}\mathcal{E}x)\gamma$ (equivalently, $\lambda x(x = x)\gamma$, abbreviated as $\mathcal{D}\gamma$) and (2) $\lambda x(\Box(\mathcal{D}\gamma \rightarrow x = \gamma))\gamma$.

SEMANTIC TABLEAUX AND NATURAL DEDUCTION FOR PREDICATE ABSTRACTION AND DESCRIPTORS

I. SEMANTIC TABLEAUX

A branch containing $\gamma \neq \gamma, i$ (for a descriptor γ) need *not* be closed. The universal and existential instantiation rules, and the identity invariance rule, may *not* be used with respect to descriptors. The identity substitution rule may be used, but only for *atomic* formulas F. New rules (γ is any descriptor):

Designator introduction	
Atomic sentence in which γ occurs, i	$\lambda xF\gamma, i$
$\gamma = b, i$	$\gamma = b, i$
(b is any constant *new* to the branch)	

Abstraction instantiation			
$\lambda xFa, i$	$\sim\lambda xFa, i$	$\gamma = a, i$ $\lambda xF\gamma, i$	$\gamma = a, i$ $\sim\lambda xF\gamma, i$
$F_x(a), i$	$\sim F_x(a), i$	$F_x(a), i$	$\sim F_x(a), i$
(a is any constant)		(a is any constant)	

To say that a descriptor 'occurs' in an atomic sentence is to say that the descriptor is one of the n terms that follow the n-place predicate. The designator introduction rule may be used only if no line equating (at world i) γ with a constant already appears on the branch. The premise of the designator introduction rule is *not* checked when the rule is used, but the (non-identity) premise of the abstraction instantiation rule *is* checked when the rule is used.

II. REPLACEMENT RULES

Name	Abbrev.	Rule	Remarks
Abstraction Replacement for Constants	ARC	$\lambda xFa \Leftrightarrow F_x(a)$	F is *any* formula.
Abstraction Replacement for Variables	ARV	$\lambda xFy \Leftrightarrow F_x(y)$	F is a formula with no bound occurrence of y.
Abstraction Replacement for Descriptors	ARD	$\lambda xF\gamma \Leftrightarrow F_x(\gamma)$	F is *atomic* with at least one free occurrence of x.

III. INFERENCE RULES

The inference rules UI, EG, EI, UG, their free and modal variants, and the identity inference rules IR, NI, and ND may *not* be used with respect to descriptors. The identity substitution rule IS may be used for formulas without modal operators. New inference rules (γ is any descriptor):

Name	Abbrev.	Rule		Remarks
Designator Introduction	DI	Atomic sentence in which γ occurs ――― $\gamma = b$	$\lambda xF\gamma$ ――― $\gamma = b$	This rule may be used only if there is no previous available line equating γ with a constant. (b is a *new* constant.)
Abstraction Instantiation	AI	$\gamma = a$ $\lambda xF\gamma$ ――― $F_x(a)$	$\gamma = a$ $\sim\lambda xF\gamma$ ――― $\sim F_x(a)$	This rule also has modal versions. Its second part is redundant given the first part of AG. (F is *any* formula.)
Abstraction Generalization	AG	$\gamma = a$ $F_x(a)$ ――― $\lambda xF\gamma$	$\gamma = a$ $\sim F_x(a)$ ――― $\sim\lambda xF\gamma$	This rule also has modal versions. Its second part is redundant given the first part of AI. (F is *any* formula.)

DEFINITE DESCRIPTIONS

I. THE GENERAL IDEA

Definite descriptions are phrases of the form ‘the so-and-so’ (e.g.: ‘the 44th U.S. President’). If one translates definite descriptions with descriptors, then some obviously valid arguments have invalid translations. For example, the argument from “Jim is the man who sang” to “Jim sang” is obviously valid, but (translating ‘the man who sang’ as γ) the translated argument from $j = \gamma$ to Sj is invalid. To remedy this problem, translate ‘the man who sang’ as $\iota x(Mx \,\&\, Sx)$ (‘the x such that x is a man and x sang’), and translate “Jim is the man who sang” as $j = \iota x(Mx \,\&\, Sx)$.

II. SYNTAX AND SEMANTICS

1. New logical symbol: ‘ι’, the *definite description quantifier*. New way of forming terms: if x is a variable and F is a formula, then ιxF is a term (informally read as ‘the x such that F’). Every occurrence of x which is free in F is bound in any formula in which ιxF occurs. To increase readability, enclose ιxF in parentheses; e.g., $T(\iota xMx)(\iota yWy)$ (“the man is taller than the woman”).

2. Given that now terms are defined by referring to formulas, and that formulas are defined by referring to terms (because, e.g., if F is a formula, x is variable, and t is *any* term—namely a variable, a constant, a descriptor, or a definite description—then λxFt is a formula), terms and formulas are defined together. An *atomic* formula can contain definite descriptions; e.g., $T(\iota xMx)(\iota yWy)$ is atomic, since it consists of a two-place predicate followed by two terms.

3. On a given interpretation, ιxF denotes object o at world w exactly if o is the only member d of the domain such that, it a is a constant denoting d at every world, then $F_x(a)$ is a sentence true at w. The object o denoted at w need not exist at w; e.g., ‘the individual who would have resulted if the sperm from which I originate had fertilized the egg from which my brother originates’.

III. TRANSLATIONS

Sentences with definite descriptions can be translated without ι, but translations with ι are often much shorter. Examples (Cxy: x creates y; Pxy: x is more powerful than y; m: Mars; v: Venus):

English	Translation with ι	Translation without ι
The creator of Mars is the creator of Venus	$\iota xCxm = \iota xCxv$	$\exists x((Cxm \,\&\, Cxv) \,\&\, \forall y((Cym \lor Cyv) \rightarrow y = x))$
The creator of Mars has the property of being possibly more powerful than the creator of Venus	$\lambda x(\Diamond Px\iota yCyv)\iota zCzm$	$\exists x(\forall y(Cym \leftrightarrow y = x) \,\&\, \Diamond\exists z(Pxz \,\&\, \forall w(Cwv \leftrightarrow w = z)))$
The creator of the creator of Mars has the property of being possibly more powerful than the creator of Venus	$\lambda x(\Diamond Px\iota yCyv)\iota zCz(\iota wCwm)$	$\exists x\exists y((\forall z(Czm \leftrightarrow z = y) \,\&\, \forall z(Czy \leftrightarrow z = x)) \,\&\, \Diamond\exists z(Pxz \,\&\, \forall w(Cwv \leftrightarrow w = z)))$

IV. NATURAL DEDUCTION

The rules that apply to descriptors apply to definite descriptions. New replacement rule:

Name	Abbr.	Rule
Description Replacement	DR	$\iota xF = a \Leftrightarrow F_x(a) \,\&\, \forall z(F_x(z) \rightarrow z = a)$, where z is a variable that does not occur in F or in the sentence in which the replacement takes place.

Example: $\forall y(\iota xCxy = g) \Leftrightarrow \forall y(Cgy \,\&\, \forall z(Czy \rightarrow z = g))$ (“God is the creator of everything”).

PHILOSOPHY 512: MODAL LOGIC

PETER B. M. VRANAS
HANDOUT 20

QUINE’S CRITIQUE OF QUANTIFIED MODAL LOGIC

I. QUINE’S CRITIQUE

0. Preliminaries. Quine attacks systems of *quantified* modal logic containing sentences that (i) have modal operators prefixed to *open* formulas (e.g., $\exists x \Box Fx$) and (ii) are not equivalent to sentences that have modal operators prefixed only to *closed* formulas (e.g., $\Box \exists x Fx$). Quine assumes that quantification is understood *objectually* and that necessity is understood as *analyticity* (a notion whose coherence he grants for the sake of argument, despite his well-known misgivings about it).

1. Quine’s first thesis is that *identity substitution fails even for constants*: the argument from $\Box(a = a)$ and $a = b$ to $\Box(a = b)$ is invalid. This is not to deny that the rule of inference IS is sound for (i.e., yields only results validated by the semantics of) certain formal systems; it is rather to say that the proposition expressed by “the sentence $a = b$ is analytic” can be (in fact, is) false even if the propositions expressed by “the sentence $a = a$ is analytic” and by $a = b$ are true. (Quine’s point is obscured by the fact that his relevant examples involve descriptions.)

2. Quine’s second thesis is that, granting an (objectual) understanding of quantification and an understanding of necessity as analyticity, *no understanding of sentences like* $\exists x \Box Fx$ *automatically follows*. For what does it mean to say that something is such that Fx holds analytically of it? Traditional accounts of analyticity apply to *sentences*, and do not even purport to explain what it is for an *open* formula to hold analytically *of an object*.

3. Quine’s third thesis is that *the obvious ways of trying to make sense of sentences like* $\exists x \Box Fx$ *do not work*.

3a. One might suggest that $\Box Fx$ holds of an object exactly if the sentence $\Box Fa$ is true, where:

- a is *any* constant denoting the object [i.e., $\forall x(x = a \rightarrow (\Box Fx \leftrightarrow \Box Fa))$ is an axiom]. But given the failure of identity substitution (Quine’s first thesis), in general $\Box Fa$ will be true for some but not all constants a denoting the object. E.g., if h (for ‘Hesperus’) and p (for ‘Phosphorus’) denote the planet Venus, $\Box(Lh \rightarrow Lh)$ is true but $\Box(Lh \rightarrow Lp)$ is false (with Lx: there is life on x), so $\Box(Lh \rightarrow Lx)$ does not hold of Venus (or, similarly, of any other object). But then on the present suggestion $\exists x \Box(Lh \rightarrow Lx)$ is false, and existential generalization (from $\Box(Lh \rightarrow Lh)$) fails.
- a is a *privileged* constant denoting the object. But how to make sense of a constant being privileged?

3b. One might alternatively suggest that $\Box Fx$ holds of an object exactly if being F is an analytic consequence of being P, where:

- P is *any* condition uniquely specifying the object [i.e., $\forall x(x = \iota y Py \rightarrow (\Box Fx \leftrightarrow \Box \forall y(Py \rightarrow Fy)))$ is an axiom]. But in general being F is an analytic consequence of being P for some but not all conditions P uniquely specifying the object. E.g., being an even prime and being the number of my hands both uniquely specify the number two, but being an even prime is an analytic consequence of the former and not of the latter, so $\Box Fx$ (with Fx: x is an even prime) does not hold of the number two (or, similarly, of any other object). But then on the present suggestion $\exists x \Box Fx$ is false.
- P is a *privileged* condition uniquely specifying the object. But how to make sense of a description being privileged? Nevertheless, Quine thinks that this suggestion is ‘the only hope’ of making sense of sentences like $\exists x \Box Fx$, and that quantified modal logic is thus

committed to *Aristotelian essentialism*: the thesis that certain ways of uniquely specifying an object are favored as somehow better revealing the 'essence' of the object. Quine concludes: "so much the worse for quantified modal logic. By implication, so much the worse for unquantified modal logic as well; for, if we do not propose to quantify across the necessity operator, the use of that operator ceases to have any clear advantage over merely quoting a sentence and saying that it is analytic."

II. SOME (POOR) RESPONSES TO QUINE

Response 1: *The development of possible-words semantics shows that one can make sense of sentences like* $\exists x \Box Fx$. Reply: 'semantics' as a mathematical theory of models yields no 'semantics' in the sense of a philosophical account of meaning and thus yields no response to Quine's claim that systems of quantified modal logic are intuitively unintelligible (unless one assumes essentialism).

Response 2: *Quantified modal logic is not committed to essentialism because no sentence expressing essentialism is derivable in the common systems.* Reply: Quine claims that quantified modal logic is committed to essentialism in the sense that essentialism is the only way of making sense of certain sentences. In Quine's words: "I've never said or, I'm sure, written that essentialism could be proved in any system of modal logic whatsoever."

III. LESSONS FROM QUINE'S CRITIQUE

The basic lesson is that *different formalisms are appropriate for conceptual (or logical) and for metaphysical necessity*: identity substitution for constants holds for the latter but not for the former, and quantifying into modal operators probably makes sense for the latter but not for the former. This lesson has yet to be widely learned.

INDEX OF REPLACEMENT AND INFERENCE RULES

Abbrev.	Name of rule	Kind of rule	Handout
Abs	Absorption	Replacement	1
AC	Actuality Conjunction	Replacement	13
AG	Abstraction Generalization	Inference	18
AI	Abstraction Instantiation	Inference	18
AN	Actuality Negation	Replacement	13
AR	Actuality Repetition	Replacement	13
ARC	Abstraction Replacement for Constants	Replacement	18
ARD	Abstraction Replacement for Descriptors	Replacement	18
ARV	Abstraction Replacement for Variables	Replacement	18
Assoc	Association	Replacement	1
CD	Constructive Dilemma	Inference	1
CE	Conjunction Elimination	Inference	1
CI	Conjunction Introduction	Inference	1
Comm	Commutation	Replacement	1
DA	Disjunctive Addition	Inference	1
DeM	De Morgan's Law	Replacement	1
DI	Designator Introduction	Inference	18
Dist	Distribution	Replacement	1
DN	Double Negation	Replacement	1
DR	Description Replacement	Replacement	19
DS	Disjunctive Syllogism	Inference	1
EG	Existential Generalization	Inference	8
EI	Existential Instantiation	Inference	8
Equiv	Material Equivalence	Replacement	1
Exp	Exportation	Replacement	1
FEG	Free Existential Generalization	Inference	10
FEI	Free Existential Instantiation	Inference	10
FQN	Free Quantifier Negation	Replacement	10
FUG	Free Universal Generalization	Inference	10
FUI	Free Universal Instantiation	Inference	10
HS	Hypothetical Syllogism	Inference	1
Impl	Material Implication	Replacement	1
IR	Identity Reflexivity	Inference	15
IS	Identity Substitution	Inference	15
I Sym	Identity Symmetry	Replacement	15
MEG	Modal Existential Generalization	Inference	13
MEI	Modal Existential Instantiation	Inference	13
MFEG	Modal Free Existential Generalization	Inference	13
MFEI	Modal Free Existential Instantiation	Inference	13
MFUG	Modal Free Universal Generalization	Inference	13
MFUI	Modal Free Universal Instantiation	Inference	13
MMP	Modal Modus Ponens	Inference	5
MN	Modal Negation	Replacement	3
MP	Modus Ponens	Inference	1
MT	Modus Tollens	Inference	1
MUG	Modal Universal Generalization	Inference	13
MUI	Modal Universal Instantiation	Inference	13
NA	Necessity of Actuality	Inference	13
NC	Negated Conditional	Replacement	1
ND	Necessity of Distinctness	Inference	15
NI	Necessity of Identity	Inference	15
NN	Necessity of non-actuality	Inference	13
QN	Quantifier Negation	Replacement	8
Taut	Tautology	Replacement	1
Trans	Transposition	Replacement	1
UG	Universal Generalization	Inference	8
UI	Universal Instantiation	Inference	8