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A New Model for Long-Term Stochastic Analysis and Prediction—Part I: Theoretical Background

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A new approach for calculating the long-term statistics of sea waves is proposed. A rational long-term stochastic model is introduced which recognizes that the wave climate at a given site in the ocean consists of a random succession of individual sea states, each sea state possessing its own duration and intensity. This model treats the sea-surface elevation as a random function of a "fast" time variable, and the time history of the spectral characteristics of the successive sea states as a random function of a "slow" time variable. By developing an appropriate conceptual framework, it becomes possible to express various probabilistic characteristics of the sea-surface elevation, which are sensible only in the fast-time scale, in terms of the statistics of sea-states duration and intensity, which is meaningful only in the slow-time scale. As an example, we study the random quantity $M_u(T)$ = "number of maxima of the sea-surface elevation lying above the level u and occurring during a long-term time period $[0,T]$." Exploiting the proposed framework, it is shown that, under certain clearly defined assumptions, $M_u(T)$ can be given the structure of a renewal-reward (cumulative) process, whose interarrival times correspond to the duration of successive sea states. Thus, using renewal theory, the complete characterization of the probability structure of $M_u(T)$ is obtained. As a consequence, the long-term probability distribution function of the individual wave height is rigorously defined and calculated. The relation of the present results with corresponding ones previously obtained is thoroughly discussed. The proposed model can be extended twofold: either by replacing some of the simplifying assumptions by more realistic ones, or by extending the model for treating the corresponding problems for ship and structures responses.

1. Introduction

AN INDIVIDUAL sea state, being a phenomenon of "moderate" duration [typically, of the order of hours (Laviel & Rio, 1987)], can be adequately modeled by considering the sea-surface elevation as a stationary stochastic process; this is the well-known short-term sea-state description, initiated through the pioneering works of Longuet-Higgins and Pierson in 1952. [See also Kinsman (1965)]. When, however, the time period of interest becomes large, many successive occurrences of individual sea states come into play, and a different stochastic model is required to predict mean and extreme values over such a period; this is the long-term sea-state description.

The short-term description of sea waves is well established and extensively developed, especially under the assumption of normality for the basic process "sea-surface elevation."[3] Similar remarks are valid for ship and structures responses within the context of linearity. [See, for example, St. Denis & Pierson (1953), Ochi & Bolton (1973), Price & Bishop (1974), Borgman (1978); Sharpkaya & Isaacson (1981), Bishop & Price (1982), Ochi (1982), Chakrabatri (1987) and references cited therein]. The key tool in this description is the spectral density function that is usually specified by means of various shape parameters, frequently some spectral moments. Using these moments, a lot of useful information concerning statistical wave characteristics can be obtained. Examples include excursion analysis (such as, mean number of crossings of a given level per unit time) and individual extreme value analysis (such as, mean number of local maxima (peaks) per unit time, and the statistics of the individual wave height) [Ochi & Bolton (1973), Price & Bishop (1974), Ochi (1982), Middleton (1960), Cramer & Leadbetter (1967)], as well as global extreme-value analysis, that is, the statistics of the global maximum (highest wave) over a time interval of given length, [Ochi (1982,1973,1981), Longuet-Higgins (1984), Naess (1984)].

In the long-term case, similar quantities should be predicted over a larger time period, for example, a month, a year, or many years. But now the situation is much more complicated since the assumption of stationarity is clearly not applicable on the sea-surface elevation. In this case, it is generally accepted that the statistics of some spectral parameters (usually of the significant wave height H_S and the mean zero-upcrossing period T_0), in conjunction with the corresponding short-term statistical results, might be used to determine the desired long-term quantities. However, the

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[3]Non-Gaussian models for the process "sea-surface elevation" have to be used in the case where hydrodynamic nonlinearities are taken into account [Longuet-Higgins (1963), Huang & Long (1980), Tayfun (1980,1981,1984), Huang et al. (1983)].

Nomenclature

a = zero-to-crest wave amplitude

$\mathbf{E}^{\beta}[\ldots]$ = ensemble average operator extended over sample space B

$f(H_S)$ = pdf of significant wave height H_S

$f(H_S,T_0)$ = joint pdf of H_S and T_0

$f(T_0)$ = pdf of mean zero-upcrossing period T_0

$f(\Delta T,\vec{\Lambda})$ = joint pdf of sea-state duration (ΔT) and intensity ($\vec{\Lambda}$)

$f(\Delta T,H_S,T_0)$ = joint pdf of ΔT, H_S and T_0

$f(\Delta T|H_S,T_0)$ = conditional pdf of ΔT for given value of H_S and T_0

$f(\Delta T|\vec{\Lambda})$ = conditional pdf of sea-state duration for a given sea-state intensity

$f_{cl}(\vec{\Lambda})$ = classical first-order pdf of the stationary stochastic process $\vec{\Lambda}(\tau)$

$f_{cl}(H_S,T_0)$ = special case of $f_{cl}(\vec{\Lambda})$

$f_{mg}(\Delta T)$ = marginal pdf of ΔT obtained by integrating $f(\Delta T,\vec{\Lambda})$

$f_{mg}(\vec{\Lambda})$ = marginal pdf of $\vec{\Lambda}$ obtained by integrating $f(\Delta T,\vec{\Lambda})$

$f_{mg}(H_S,T_0)$ = special case of $f_{mg}(\vec{\Lambda})$

$f_{sd}(H_S,T_0)$ = empirical joint pdf of H_S and T_0 (scatter diagram)

$\mathcal{F}$ = operator realizing the mapping $\eta(t) \rightarrow \Xi(i)$

$\mathcal{F}_1$ = operator realizing mapping $\eta(t) \rightarrow \vec{\Lambda}(\tau)$

$\mathcal{F}_2$ = operator realizing mapping $\vec{\Lambda}(\tau) \rightarrow \Xi(i)$

$\mathcal{F}^{-1}$ = generalized inverse of $\mathcal{F}$; see Section 4

$g_m(\vec{\Lambda})$ = auxiliary "density" function; see equation (20)

$g_1(\sigma,T_0,\varepsilon)$ = special case of $g_m(\vec{\Lambda})$

$g_1(H_S,T_0)$ = marginal "density" obtained by integrating $g_1(\sigma,T_0,\varepsilon)$

H = crest-to-trough wave height

H_S = significant wave height

$M(u;\Delta T,\vec{\Lambda})$ = mean number of peaks above level u occurring during a time interval ΔT in a stationary sea state of intensity $\vec{\Lambda}$

$M_u([T_1,T_2];\beta)$ = [or $M_u(T_1,T_2;\beta)$] number of peaks (local maxima) of sea-surface elevation lying above level u and occurring in a (long-term) time interval $[T_1,T_2]$

$M_u(T;\beta) = M_u([0,T];\beta)$

$P_L(H)$ = long-term CDF of individual crest-to-trough wave height

$P_L(a)$ = long-term CDF of individual zero-to-peak wave amplitude

$P_L^+(a)$ = conditional long-term pdf of individual zero-to-peak wave amplitude, given that the peak is nonnegative

$\mathrm{PR}[A]$ = probability of occurrence of event A

$R(H;H_S)$ = Rayleigh CDF

$r(H;H_S)$ = Rayleigh pdf

$S(\omega),S_i(\omega)$ = spectral density function (spectrum) of a sea-state

$S(\omega;\tau_*,\beta_*)$ = short-term spectrum generated by $\Lambda(\tau_*,\beta_*)$

t = "short-term" or "fast" or "fine" time variable (time variable of fast-time scale)

T_0 = mean wave period between zero upcrossings

$\hat{T}_0,\hat{T}_1,\hat{T}_2,\hat{T}_3$ = characteristic times; see Section 2

T_b,T_{bi} = time instants representing beginning of a sea state

T_e,T_{ei} = time instants representing end of a sea state

T_{RI} = recording interval: time between two successive measurements of sea-surface elevation

T_{RP} = recording period: time the recording instrument remains in operation

$W(u;1,\vec{\Lambda}) = M(u;1,\vec{\Lambda})$, that is, mean number of peaks per unit (fast) time occurring in a sea-state of intensity $\vec{\Lambda}$

Greek letters

B = sample space of the stochastic process "long-term time history of sea-surface elevation at a given location"

$\Gamma = \Gamma(\tau_*,\beta_*)$ = sample space of stochastic process "short-term time history of sea-surface elevation generated by $\vec{\Lambda}(\tau_*,\beta_*)$"

β,γ = choice variables ranging through sample spaces B and Γ, respectively

$\Delta T,\Delta T_i$ = duration of a sea state

$\Delta\vec{\Lambda}$ = vector of increments of $\vec{\Lambda}$: $\Delta\vec{\Lambda} = (\Delta\Lambda_1, \Delta\Lambda_2, \ldots, \Delta\Lambda_L)$

ε = broadness coefficient of the spectrum

$\eta(t)$ = sea-surface elevation

$\eta(t,\beta)$ = sample function of stochastic process "sea-surface elevation at a given location"

$\eta_H(t,\beta;\{\gamma_i\})$ = "hindcasted' sample function of the process "sea-surface elevation generated by a sample function $\vec{\Lambda}(\tau,\beta)$ by means of the operator $\mathcal{F}^{-1}$": $\eta_H(t,\beta;\{\gamma_i\}) = \mathcal{F}^{-1}(\vec{\Lambda}(\tau,\beta))$

$\eta^{ST}(t;\tau_*,\beta_*)$ = short-term stochastic process generated by $\vec{\Lambda}(\tau_*,\beta_*)$

$\vec{\Lambda}(\tau)$ = time history of spectral parameters of sea states occurring at a given location. For example, $\vec{\Lambda}(\tau) = (H_S(\tau),T_0(\tau))$

$\vec{\Lambda}(\tau,\beta)$ = sample path of stochastic process "time history of spectral parameters of the sea states occurring at a given location"

$\vec{\Lambda}_i,\vec{\Lambda}_*$ = spectral parameters (intensity) of a sea state

$\mu_{m,\Delta T}$ = mth-order moment of random variable ΔT

$\mu_{m,M}$ = mth-order moment of random variable $M = M(u;\Delta T,\vec{\Lambda})$

$\mu_{\Delta T},\mu_M$ = mean values of random variables ΔT and $M = M(u;\Delta T,\vec{\Lambda})$

$\Xi(i)$ = sequence of successive individual sea states occurring at a given location, $\Xi(i) = \{\Delta T_i,\vec{\Lambda}_i\}$

$\Xi(i,\beta)$ = sample path of stochastic process "sequence of successive individual sea states occurring at a given location," $\Xi(i,\beta) = \{\Delta T_i(\beta),\vec{\Lambda}_i(\beta)\}$

$\rho_{\Delta T,M}$ = correlation coefficient of random variables ΔT and $M = M(u;\Delta T,\vec{\Lambda})$

$\sigma^2_{\Delta T}$, σ^2_M = variances of random variables ΔT and $M = M(u;\Delta T,\vec{\Lambda})$

τ = "long-term" or "slow" or "coarse" time variable (time variable of slow-time scale)

Acronyms

pdf = probability density function

CDF = cumulative distribution function

i.i.d. = independently and identically distributed existing stochastic models do not, generally, properly treat the non-stationarity of the sea-surface elevation. Moreover, the underlying assumptions concerning stationarity/non-stationarity and stochastic independence are not explicitly stated, so that the planning engineer cannot identify the circumstances under which a given model is valid. Other inadequacies of the currently used long-term stochastic models will be pointed out subsequently.

Our objective in this paper is to develop a new long-term stochastic model taking explicitly into account the fact that the wave climate at a given site in the ocean consists of a random succession of individual sea states, each sea state possessing its own duration and spectral characteristics. A significant effort is made to isolate and explicitly state all the assumptions used in constructing the model or in simplifying the results. This has become possible by developing a detailed conceptual framework for the non-stationary process "sea-surface elevation over long-term time periods."

In order to explain our point of view and clarify the relation of our model with the existing ones, we felt it necessary to begin by viewing the literature on the subject. The first works dealing with long-term stochastic analysis and prediction came up in the late 50's. Their main concern was the calculation of the long-term probability density function of the wave-induced ship responses, with emphasis in structural loading and structural responses [Jasper (1956) and Bennet (1958,1959)]. Since, however, the method of analysis for calculating the long-term probability distributions of ship responses and of the wave height is actually the same, under the assumption of linearity, we shall restrict our attention in this section (and, in fact, in this paper) only to the latter case.

In the early works on the subject, use was made of the joint probability density function $f(H,H_S)$, where H is the individual crest-to-trough wave height, and H_S is the significant wave height of the corresponding sea state. No attempt was made, however, to define directly the event which has probability $f(H,H_S)\Delta H \Delta H_S$.[4] Instead, the density $f(H,H_S)$ was defined indirectly, by using a conditional probability argument, as follows

$$f(H,H_S) = f(H|H_S)f(H_S) \tag{1}$$

where $f(H|H_S)$ is the conditional probability density function of the individual wave height H given that the significant wave height is equal to H_S, and $f(H_S)$ is the probability density function (subsequently abbreviated as pdf) of the significant wave height in the considered sea area. Then, the long-term cumulative distribution function of the individual crest-to-trough wave height, denoted by $P_L(H)$, is obtained by integrating over all possible H_S ("all possible short-term sea states"). Applying this approach and approximating $f(H|H_S)$ by the Rayleigh pdf

$$r(H;H_S) = 4H(H_S)^{-2} \exp[-2(H/H_S)^2]$$

one obtains

$$P_L(H) = \mathbf{P}\textsc{r}[\text{wave height} \leq H] = \int_0^\infty R(H;H_S)\, f(H_S) dH_S \tag{2}$$

where $R(H;H_S)$ is the cumulative distribution function (subsequently abbreviated as CDF) corresponding to $r(H;H_S)$. More elaborate models have also been suggested, obtained by further conditioning with respect to wave period or to weather conditions (for example, wind force). [Bennet (1958,1959), Nordenstrom (1964,1969,1973), Band (1966), and Lewis (1967)].[5] Variants of this approach have also been followed by many other authors for long-term seakeeping calculations. [Fukuda (1970), Loukakis & Grivas (1980), Lewis & Zubaly (1981), Stiansen & Chen (1982), Hughes (1983), and Chilo et al. (1986)]. Recently, Spouge (1985,1986) presented a lucid formulation of the long-term stochastic ship seakeeping performance problem along these lines. He extended the above described approach to include a probabilistic description of ship's mission variability, weather avoidance tendency, and operational sea-man-ship characteristics.

Despite its broad acceptance by the naval architecture community, and its usefulness for comparative studies of different designs, the above definition of the long-term probability distribution is questionable, at least in the following sense. It is not clear whether or not the CDF $P_L(H)$ obtained by equation (2) is compatible with the usual "statistical" definition of the notion of probability, based on the relative frequency concept. In other words, it is not clear whether or not $P_L(H)$ satisfies the relation

$$P_L(H) = \frac{\text{No. of waves with a height smaller than } H}{\text{No. of all waves appearing in same period}} \tag{3}$$

provided the counting period is a long-term one. Note that equation (3) is always (tacitly) assumed to be valid in any application of $P_L(H)$. Nevertheless, in the light of a more careful analysis, it can be proven that, generally, equation (3) is not satisfied by $P_L(H)$, if the latter is defined through equation (2). [See the comments below, after equation (5).]

The decisive step towards a satisfactory definition of the long-term CDF $P_L(H)$ was made by Battjes in 1970 [see also Bishop & Price (1982) and Battjes (1970)], who defined $P_L(H)$ by means of the relation

$$P_L(H) = \frac{\mathbf{E}[\text{No. of waves with a height smaller than } H]}{\mathbf{E}[\text{No. of all waves appearing in same period}]} \tag{4}$$

where $\mathbf{E}[\ldots]$ denotes the mean-value operator, and the counting period is again a long-term one.[6] Thus, the calculation of the long-term probability distribution is reduced to the calculation of the mean number of waves meeting some condition (namely, having a height less than a certain level) and appearing in a long-term time period. Calculating this mean number, Battjes was able to express $P_L(H)$ as follows

$$P_L(H) = \frac{1}{\mathbf{E}[1/T_0]} \int_0^\infty \int_0^\infty \frac{R(H;H_S)}{T_0} f(H_S,T_0) dH_S dT_0 \tag{5}$$

where $f(H_S,T_0)$ is the joint pdf of H_S and T_0, and $\mathbf{E}[1/T_0]$ is the long-term expected number of waves per unit time. Note that, if we assume that H_S and T_0 are statistically independent, that is, $f(H_S,T_0) = f(H_S)f(T_0)$,[7] then equation (5) is reduced to the previous result (2). This fact shows that the CDF $P_L(H)$, as defined by equation (2), does not satisfy relation (3), whenever H_S and T_0 are not independent. Battjes's method has been subsequently used and improved by Ochi (1978*a*,*b*,1982) and Ochi & Chang (1978), who extended it

[4]In our point of view, such an event does not seem well-defined, since H and H_S belong to different "stochastic levels"; see below, in Section 4.

[5]An informative survey of these works has been presented by Ochi and Bolton (1973) in their review article.

[6]The exact meaning and significance of the assumption that the counting period is a long-term one will be made clear subsequently, in Section 6.

[7]Throughout this work we shall use the same symbol to denote a joint pdf and its marginal pdfs. However, since the arguments will always be written, it is not likely to cause any confusion.

for making long-term predictions of the responses of ships and floating structures in waves [see also Hughes (1983)].

Battjes's approach is both sound and fruitful. In our opinion, equation (4) is a quite reasonable way to define the long-term CDF of the individual wave height, provided the counting time is a long-term period. There are, however, questions which require further study if we wish to achieve a clear and reliable long-term probabilistic description of sea waves and structure responses. Let us mention some of them here:

- Which exactly are the assumptions ensuring the validity of equation (5)?
- It is well-known that the various sea states last for different time periods. Is the effect of this sea-state duration variability on the long-term calculations significant?
- It seems very likely that some statistical dependence between successive sea states exists [Laviel & Rio (1987), Labeyrie (1990)]. How could this dependence be incorporated in a long-term stochastic model?

The construction of a stochastic model capable to treat such questions is the main objective of the present work. In accordance with equations (3) and (4), the main concern of this model will be the determination of the statistics of the random variable "number of waves which appear in a long-term time period and meet some prespecified condition."

Another interesting and fruitful approach to the statistical prediction of wave characteristics (more exactly, the prediction of extreme values) was introduced by Borgman (1973,1972) and Borgman & Resio (1982). [See also Krogstad (1985)]. Borgman was apparently the first author who explicitly took into acount the non-stationarity of the process "sea-surface elevation over large-time intervals." To handle this difficulty, he assumed that the spectral parameters can be considered as continuous functions of time,[8] and he divided the time period of interest into small subintervals so that the sea state in each of them is approximately constant. Then, assuming that successive local maxima are statistically independent, he was able to obtain an expression for the CDF of extreme values of individual wave height which is applicable for arbitrary periods of time. Note, however, that this theory presupposes that the time history of the spectral parameters is known over the whole time period of interest. Thus, although this theory treats stochastically each individual sea state, it handles deterministically the succession of the various sea states, occurring at a given site, that is, the non-stationarity of the phenomenon.

In 1977 Battjes, in his excellent review paper, took Borgman's conception (that the spectral parameters can be considered as ordinary functions of time), to what seems to be a sound and promising basis for a long-term stochastic analysis and prediction of sea waves and, actually, of many other similar phenomena. He assumed that the time history of the spectral parameters can be considered as a realization of a stochastic process different from the stochastic process of the sea-surface elevation. This is, in fact, the fundamental idea underlying the present work.

The content of the present paper can be summarized as follows: In Section 2 we introduce the characteristic times related to the complex (short-term and long-term) phenomenon, and we define the two time scales t (the "short-term" or "fast" or "fine" time scale) and τ (the "long-term" or "slow" or "coarse" time scale) used in the analysis. In Section 3 we define various notions of fundamental importance for our model, including the individual sea state, its duration and its intensity. It is also pointed out there that a given sample path of the long-term sea-surface elevation $\eta(t)$ gives rise to a time history of its spectral characteristics $\vec{\Lambda}(\tau)$ [for example, $\vec{\Lambda}(\tau) = (H_S(\tau), T_0(\tau))$], which, in its turn, can produce a sequence $\Xi(i) = \{\Delta T_i, \vec{\Lambda}_i\}$ representing the succession of individual sea states, where ΔT_i and $\vec{\Lambda}_i$ denote the intensity and the duration of the ith sea state, respectively.

In Section 4 a stochastic point of view is adopted, and $\eta(t)$, $\vec{\Lambda}(\tau)$, and $\Xi(i)$ are all viewed as stochastic processes. Having defined these processes, we are in a position to clearly formulate the main assumptions on which the proposed model is based. From the physical point of view, the most fundamental assumption used in this work is the first-order seasonal stationarity of the stochastic sequence $\Xi(i)$, which can be expressed as follows

> To each ocean site and season[9] we associate a sea-state population and we assume that there exists a time-invariant (seasonal) pdf $f(\Delta T, \vec{\Lambda})$ describing the first-order statistics of this population.

This stationarity assumption restricts but does not suppress the non-stationarity of the sea-surface elevation itself, since the latter is treated as a succession of different individual sea states, each one having its own duration and spectral characteristics. It should be noted, however, that other assumptions concerning the stochastic nature of the long-term process might be also adopted. For example, $\Xi(i)$ might be given a Markovian structure, or $\vec{\Lambda}(\tau)$ can be considered as a periodically correlated process. Such choices would strongly complicate the analysis and will not be examined here.

The basic difficulty in long-term analysis and prediction is that we have to calculate probabilistic characteristics of sea waves defined on the process $\eta(t)$, having at our disposal statistical data concerning the process $\Xi(i)$. To overcome this difficulty, we explicitly construct the procedures (operators) $\mathcal{F}$ and $\mathcal{F}^{-1}$, relating the sample paths of the stochastic processes $\eta(t)$ and $\Xi(i)$

$$\Xi(i) = \mathcal{F}(\eta(t)) \quad \text{and} \quad \eta(t) = \mathcal{F}^{-1}(\Xi(i)) \qquad (6a,b)$$

As is expected, both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are multiple-valued with respect to the sample paths themselves, but carry over, in an adequate way, the necessary statistical information from the one stochastic process to the other. The main idea here is to use relation (6*b*) in order to express probabilistic characteristics of the sea-surface elevation $\eta(t)$ with the aid of the statistics of the process $\Xi(i)$, which can be taken as known.

This principle is applied in Section 5 for obtaining the probabilistic characteristics of the random quantity $M_u(T)$ = "number of maxima (peaks) of the sea-surface elevation lying above the level u and occurring during a long-term time period $[0,T]$." It is recognized that the conceptual framework constructed in Sections 3 and 4, in conjunction with the above stated seasonal stationarity assumption, and an independence assumption between the successive sea states, permits us to consider $M_u(T)$ as the up-to-time T "accumulated cost" of a renewal-reward (cumulative) process, whose interarrival times are the durations of the sea states. Thus, the arsenal of the renewal (renewal-reward) theory becomes available, and the complete characterization of the probabilistic structure of $M_u(T)$ is made possible. It turns out that $M_u(T)$ is normally distributed with mean value and variance explicitly obtained in terms of the first-order sta-

[8] As will be explained subsequently, in Section 3, this assumption is meaningful only if we consider the spectral parameters as functions of a time variable which is "slower" than the time variable used in describing the process "sea-surface" elevation.

[9] In this statement the word "season" should be merely thought as a certain definite part of the calendar year. It may be a month, a calendar season or even the whole year, depending on the type of application we have in mind and on the available wave data.

tistics of the stochastic process $\Xi(i)$. Moreover, it is shown that the mean value of $M_u(T)$ can be also expressed through the first-order statistics of the continuous-time process $\vec{\Lambda}(\tau)$. The results of Section 5 are then used in Section 6 to rigorously define and calculate the long-term CDFs of the individual wave amplitude and wave height $P_L(a)$ and $P_L(H)$, respectively. In Section 7 we prove and discuss an interesting and somewhat surprising relation between the first-order pdfs of the processes $\Xi(i)$ and $\vec{\Lambda}(\tau)$. The relation of the present results with the corresponding ones obtained by Battjes is discussed at the end of Section 5 and in Section 6. The paper is concluded by Section 8, where the whole model is summed up in a rather detailed yet non-formal way.

2. Short- and long-term time scales

In analyzing the process "sea-surface elevation at a given location" over long-term periods,[10] it is of fundamental importance to distinguish various time scales [Battjes (1977)]. This enables us to introduce a hierarchy structure in the process, and to focus our attention separately on its various particular aspects. Before introducing the two principal time scales, it is advisable to describe some characteristic times (time lengths) related to the whole process under consideration. These are the following:

(*a*) A time length $\hat{T}_0$ comparable to a mean period of wind waves.

(*b*) A time length $\hat{T}_1$ during which the sea state can be considered statistically stationary, and which should be sufficiently large for short-term sampling purposes.

(*c*) A time length $\hat{T}_2$ comparable to the time needed for the mean statistical characteristics of the sea state to change in a significant percentage, say, 30 percent.

(*d*) A time length $\hat{T}_3$ containing a great number of individual sea states[11] so that it can be considered sufficiently large for long-term sampling purposes.

Other characteristic times can be also distinguished, but they are of no interest for the present work.

In order for a two-level (short-term/long-term) analysis of the process "sea-surface elevation" to be meaningful, the following order-of-magnitude assumptions are necessary

$$\frac{\hat{T}_0}{\hat{T}_1} \ll 1 \qquad \frac{\hat{T}_1}{\hat{T}_2} \ll 1 \qquad \text{and} \qquad \frac{\hat{T}_2}{\hat{T}_3} \ll 1 \qquad (7a,b,c)$$

The first assumption means that many cycles of the process "sea-surface elevation" are contained in $\hat{T}_1$, and it is necessary (although not sufficient) for the validity of the ergodicity hypothesis in short-term analysis (first-level ergodicity). The second assumption ensures that the statistical characteristics vary slowly with respect to the time $\hat{T}_1$. This is necessary for the consistency of the piecewise stationarity (short-term or first-level stationarity) of the process, which is generally taken for granted. The third assumption is necessary (but not sufficient) for the validity of the ergodicity hypothesis in long-term analysis (second-level ergodicity), which will be used in Section 4.

Existing experience shows that $\hat{T}_0$ can be assumed of the order of seconds, $\hat{T}_1$ of the order of minutes or tens of minutes, and $\hat{T}_2$ of the order of hours. Accordingly, the inequalities (7*a*,*b*) can be considered generally valid. The inequality (7*c*) should be considered rather as a condition defining a typical long-term time period $\hat{T}_3$, sufficiently large for long-term sampling purposes.

[10]That is, periods extending over many years.

[11]The notion of an individual sea state will be precisely defined in the next section. Here it suffices to consider this notion in some intuitive (or meteorological) sense.

The two time scales t and τ needed in the present work can now be defined by means of the following inequalities:

$$\frac{\text{unit of } t}{\hat{T}_1} \ll 1 \qquad \text{and} \qquad 1 < \frac{\text{unit of } \tau}{\hat{T}_1} < \frac{\hat{T}_2}{\hat{T}_1} \qquad (8a,b)$$

That is, the unit of t is comparable to $\hat{T}_0$, while the unit of τ is comparable to $\hat{T}_2$. The first time scale, in which the oscillation of the sea-surface elevation is clearly "visible" and its statistical characteristics can be considered constant, will be referred to as the short-term or fast or fine time scale. Note that, in the present work, we are interested in counting events visible in this time scale (such as the number of peaks above a given level u), over time intervals $[\tau_1,\tau_2]$ of the order of $\hat{T}_3$. Accordingly, it will be necessary to partition these intervals into smaller ones, comparable with or smaller than $\hat{T}_1$, and count the events in each interval sequentially.

The second time scale, in which the oscillations of the sea-surface elevation are "invisible," while the evolution of its statistical characteristics is sensible, will be referred to as the long-term or slow or coarse time scale.

In the next two sections a procedure will be developed enabling us to model the process "sea-surface elevation over long-term time periods" as a stochastic process of the two time variables t and τ.

3. Sea states and their duration

Consider a long-term record of the sea-surface elevation $\eta(t)$, at a given location in the ocean. The purpose of this section is to contrive a way to divide such a record into a succession of "individual sea states," that is, successive parts of it over each of which the statistical sea-state characteristics remain (approximately) constant. Our main concern is to present a flexible definition of an individual sea state, which will be both conceptually clear and practically useful in long-term analysis and prediction. We should, however, have in mind that, in practice, a complete long-term record of the sea-surface elevation is almost never available. Accordingly, before proceeding to define sea states and their durations, it seems advisable to take a look at the existing sources of wave data.

The three principal sources of long-term wave data are instrumental records, hindcasted time-series of spectral parameters, and visual observations [Battjes (1977), Muir & El-Shaarawi (1986)]. Instrumental records are not generally continuous. Recording instruments usually work intermittently. They are activated every, say, 3 hr (the recording interval T_{RI}), and remain in operation for, say, 20 min (the recording period T_{RP}).[12] Considering that each 20-min record is a part of a realization of a stationary and ergodic stochastic process (this is the usual short-term randomization of sea waves), we obtain, after appropriately processing each record, a sequence of spectral density functions $S_i(\omega)$, from which the (vector-valued) sequence of spectral characteristics $\vec{\Lambda}_i = (\Lambda_{1i},\Lambda_{2i},\ldots,\Lambda_{Li})$ can be easily calculated. Switching now to the slow (coarse) time scale τ, we can assume that the sequence of measured spectral parameters $\{\vec{\Lambda}_i\}$ defines, with the aid of some interpolation procedure (for example, linear or spline interpolation), a continuous (vector-valued) function of time $\vec{\Lambda}(\tau) = (\Lambda_1(\tau),\Lambda_2(\tau), \ldots, \Lambda_L(\tau))$ [Borgman (1973) and Battjes (1977)]. Summarizing, we can say that we have defined a filter (operator) $\mathcal{F}_1$, which, to each long-term sea-surface elevation record $\eta(t)$, associates the time history of its spectral parameters $\vec{\Lambda}(\tau) = \mathcal{F}_1(\eta(t))$.

Time histories of spectral parameters $\vec{\Lambda}(\tau)$ can be also directly produced by using various hindcasting techniques

[12]For consistency it is required $T_{RP} = 0(\hat{T}_1)$ and $\hat{T}_1 < T_{RI} < \hat{T}_2$.

currently available [Lazanoff & Stevenson (1975), Cardone et al (1976), Resio (1981), Bales et al (1982), Golding (1983)]. The major advantage of hindcasted wave data is that they can be easily obtained for practically all the offshore sites on the globe,[13] and that they can be extended over large (multiyear) time periods without awaiting the years to pass. This fact is of fundamental importance for the quick collection of the great amount of wave data needed for reliable long-term calculations.

Visual observations from ocean weather ships or voluntary observing ships do not seem very useful for hindcasting the evolution of the sea state at a given site, since the observers were moving along the ship's course. However, visual observations from light ships and gas or oil platforms could, in principle, be used for this purpose. In any case, the authors are not aware of any pertinent publication.[14]

Having the above in mind, we shall proceed to define the notion of an individual sea state and its duration, based on the time history of a given set of spectral parameters $\vec{\Lambda}(\tau) = [\Lambda_1(\tau),\Lambda_2(\tau), \ldots, \Lambda_L(\tau)]$. The notion of sea-state duration described below can be considered as a generalization of the corresponding notion used by Bales et al (1982,1981), which is based exclusively on the time history of the significant wave height.

Let us consider $\vec{\Lambda}(\tau)$ as a curve (path) in the L-dimensional Euclidean space $\mathbb{R}^L$ with axes $\Lambda_1,\Lambda_2, \ldots, \Lambda_L$. Consider also the "planes"

$$\Lambda_1 = k_1\Delta\Lambda_1, \qquad k_1 = 1,2,3, \ldots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Lambda_L = k_L\Delta\Lambda_L, \qquad k_L = 1,2,3, \ldots$$

where $\Delta\Lambda_1,\Delta\Lambda_2, \ldots, \Delta\Lambda_L$ are prespecified increments of the quantities $\Lambda_1,\Lambda_2, \ldots, \Lambda_L$, respectively. An example for the case $L = 2$ is depicted in Fig. 1. All these "planes"[15] taken together define a grid (covering the positive cone of $\mathbb{R}^L$), which will be denoted by $G(\vec{\Lambda},\Delta\vec{\Lambda})$, $\Delta\vec{\Lambda}$ being a shorthand notation for the vector of increments $(\Delta\Lambda_1,\Delta\Lambda_2, \ldots, \Delta\Lambda_L)$. It is convenient to give first the definition of the transition from one sea state to another. This will be considered to occur whenever the path $\vec{\Lambda}(\tau)$ intersects the grid $G(\vec{\Lambda},\Delta\vec{\Lambda})$. Then, an individual sea state is defined as the set of values $(T_b,T_e,\vec{\Lambda}_*)$, where T_b and T_e are the time instants corresponding to two successive transitions and representing, respectively, the beginning and the end of the sea state, while $\vec{\Lambda}_* = (\Lambda_{1*},\Lambda_{2*}, \ldots, \Lambda_{L*})$ is a value of the spectral parameters vector representing the nearly constant $\vec{\Lambda}(\tau)$ during the time interval $[T_b,T_e]$. For example, $\vec{\Lambda}_* = \vec{\Lambda}(\tau_*)$ for some $\tau_* \in (T_b,T_e)$. See Fig. 1. The time length $\Delta T = T_e - T_b$ will be called the duration of the sea state, while the value $\vec{\Lambda}_*$ will be called the intensity of the sea state. In most cases we are not interested in the time instants T_b and T_e, but only in the sea-state duration ΔT. Then, a sea state will be represented by the reduced set of values $(\Delta T,\vec{\Lambda}_*)$.

Although the above described procedure of defining the intensity and duration of a sea state can be applied for any prespecified set of spectral parameters, it can be directly realized graphically only when the dimension L of the vector $\vec{\Lambda}(\tau)$ is equal to 1,2 or 3. (This actually includes all practically useful cases.) We now describe an alternative way of pictorial realization of the above definition of sea states, which works for arbitrary L. The key idea is to plot each component $\Lambda_\ell(\tau)$, $l = 1,2, \ldots, L$, separately, and draw the "partial" grids $G(\Lambda_\ell,\Delta\Lambda_\ell)$, consisting of the level lines $\Lambda_\ell = k_\ell\Delta\Lambda_\ell$, $k_\ell = 1,2,3, \ldots$, on the corresponding planes (τ,Λ_ℓ). Then, a transition occurs whenever any of the curves $\Lambda_\ell(\tau)$, $l = 1,2, \ldots, L$, intersects the "partial" grid of its plane $G(\Lambda_\ell,\Delta\Lambda_\ell)$. Clearly, in this case the transition points are scattered in the L different (τ,Λ_ℓ)-planes. Thus, to find correctly the succession of sea states as defined above, we have to collect all transition instants on a single τ-axis. In Fig. 2 we use this graphical representation to find the transition instants corresponding to the hypothetical sea-state evolution shown in Fig. 1. The same increments $\Delta\Lambda_1$ and $\Delta\Lambda_2$ have been used in the two figures.

In any case, it should be emphasized that the individual sea state and its duration, as defined above, are strongly dependent both on the set $\vec{\Lambda}$ of spectral parameters used for defining the intensity of the sea state, and on the grid $G(\vec{\Lambda},\Delta\vec{\Lambda})$ used for defining the transition levels. Whenever this dependence should be explicitly stated, we shall speak of a $(\vec{\Lambda},[\vec{\Lambda}_i,\vec{\Lambda}_j))$ sea state or a $\Delta T(\vec{\Lambda},[\vec{\Lambda}_i,\vec{\Lambda}_j))$ sea state duration, where $[\vec{\Lambda}_i,\vec{\Lambda}_j)$ is the representative interval (cell) of the grid $G(\vec{\Lambda},\Delta\vec{\Lambda})$. For example, an $(H_S,[4m,5m))$ sea state is a sea state defined only on the basis of the significant wave height, in which H_S is kept within the range $4m \leq H_S < 5m$.

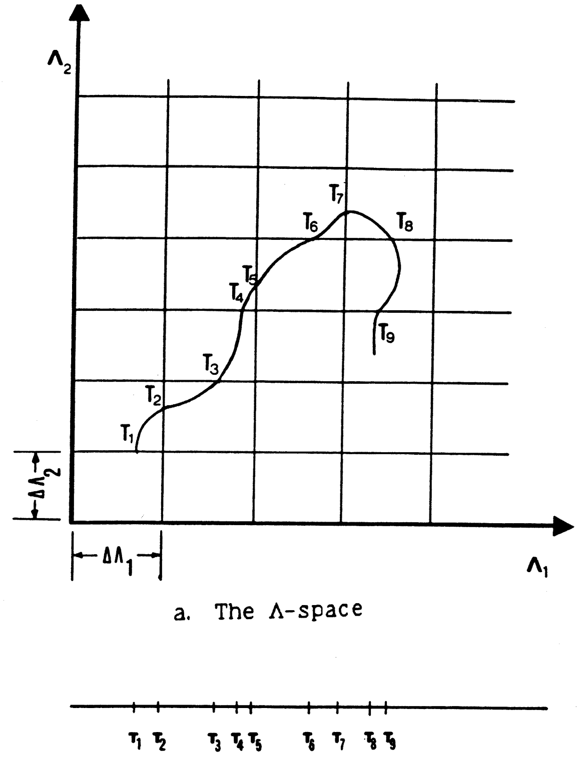

Fig. 1 Sea-state evolution as represented in a two-dimensional $\vec{\Lambda}$-space. (*a*) The curve $T_1,T_2 \ldots T_9$ represents the evolution of the sea state, while the points $T_1,T_2, \ldots, T_9$ represent the transitions from one sea state to another. (*b*) The points $\tau_1,\tau_2, \ldots, \tau_9$ on the time axis correspond to the instants of transitions. Note that the time intervals $\tau_i\tau_{i+1}$ are not in general related to the length of the corresponding arcs T_iT_{i+1}

[13]Several organizations possess currrently running global hindcast programs, among them we mention the U.S. Navy and the Meteorological Office, England.

[14]Recently at the U.K. Meteorological Office, an effort was initiated to investigate a relative question, namely, the possibility of inferring sea-state-duration statistics from visual observations. However, "this work is still in its early stages and will be some time before any results are forthcoming" [personal communication (G. A. Athanassoulis)].

[15]The "planes" are points if $L = 1$, straight lines if $L = 2$, planes if $L = 3$, and "hyperplanes" if $L > 3$.

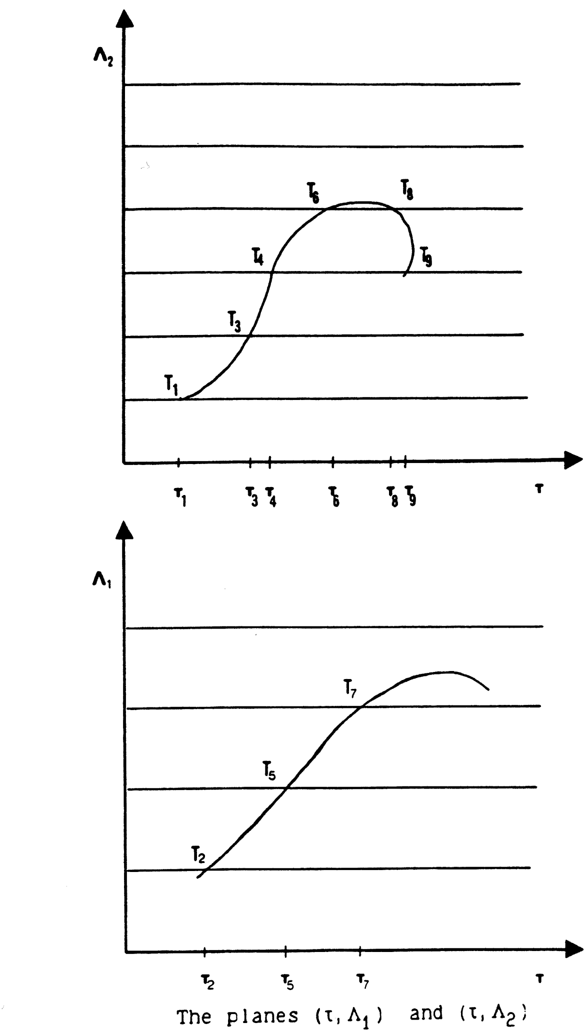

Fig. 2 Sea-state evolution as represented by means of the two curves $\Lambda_1(\tau)$ and $\Lambda_2(\tau)$. The set of transition points is now separated into Λ_1-transitions (T_2,T_5,T_7) and Λ_2-transitions $(T_1,T_3,T_4,T_6,T_8,T_9)$. The full set of transition points, that is, the (Λ_1,Λ_2)-transitions, are recovered by collecting all transition instants on the same time axis. See Fig. 1(*b*)

The above definition of a sea state also defines an operator $\mathcal{F}_2$, which, to each time history $\vec{\Lambda}(\tau)$ of a given set of spectral parameters, associates the sequence $\Xi(i) = \{\Xi_i\} = \{(T_{bi},T_{ei},\vec{\Lambda}_i)\}$[16] of successive individual sea states, where $T_{b,i+1} = T_{ei}$. In symbols, $\Xi(i) = \mathcal{F}_2(\vec{\Lambda}(\tau))$. Whenever the time instants T_{bi} and T_{ei} are of no value for us, the sequence of sea states $\Xi(i)$ will be considered defined in the following slightly different form: $\Xi(i) = \{\Xi_i\} = \{(\Delta T_i,\vec{\Lambda}_i)\}$.

An alternative way of realizing the operator $\mathcal{F}_2$ has been recently proposed by Laviel and Rio (1987), who model the transition from one sea state to another as a hypothesis-testing statistical problem. Undoubtedly, this is a more versatile and realistic way of obtaining the sequence of successive sea states in practice, when $\vec{\Lambda}(\tau)$ is derived by direct meaurements of the sea-surface elevation $\eta(t)$, in which case random instrumental errors are present. On the other hand, the simpler procedure presented above is the only possible one when the time series $\vec{\Lambda}(\tau)$ comes from hindcasting numerical models. Note, however, that the exact way in which the sequence $\Xi(i)$ is obtained is of no particular importance for the theoretical construction of our long-term stochastic model.

From the sequence $\Xi(i) = \{(T_{bi},T_{ei},\vec{\Lambda}_i)\}$ we can, in an obvious way, produce a step-function approximant of the continuous function $\vec{\Lambda}(\tau)$. Within the accuracy requirements of the present analysis, we can consider that the function $\vec{\Lambda}(\tau)$ can be retrieved from this step function, by using again some interpolation procedure. Accordingly, between time histories of spectral parameters $\vec{\Lambda}(\tau)$ and sea-state sequences $\Xi(i)$ there exists a one-to-one correspondence. As a consequence, $\vec{\Lambda}(\tau)$ and $\Xi(i)$ can be considered equivalent, and subsequently we shall refer to either of them according to our needs.

Now, we can combine the two operators $\mathcal{F}_1$ and $\mathcal{F}_2$, obtaining the composite operator $\mathcal{F} = \mathcal{F}_2\mathcal{F}_1$, which realizes the mapping $\Xi(i) = \mathcal{F}(\eta(t))$. Thus, we have achieved our objective for this section, which was to contrive a way to divide a given sea-surface elevation record into a succession of individual sea states.

Before concluding this section it is advisable to compare our definition of sea states and their duration with another one often encountered in the current literature of ocean and coastal engineering. A sea state of level H'_S is defined [after Draper (1966)] as an excursion of $H_S(\tau)$ above the level H'_S, and its duration is defined as the time interval between an up-crossing and the successive down-crossing of this level. [See also Battjes (1970), Houmb (1971), Draper (1976), Vik & Houmb (1976), Graham (1982), Kuwashima & Hogben (1986)]. It is evident that this threshold-based definition of sea states puts together sea states with essentially different spectral characteristics, leading to sea-state durations much greater than ours. Although this definition is useful for various types of problems such as constructional and operational planning, it is inadequate for our needs, as will be clearly seen below. This is why we have introduced our grid-based definition of a $(\vec{\Lambda},[\vec{\Lambda}_i,\vec{\Lambda}_j))$ sea state and its duration. On the other hand, it is interesting to note that the threshold-based definition of a sea state can be formally obtained from our definition if we set $\vec{\Lambda} = H_S$ and $[\vec{\Lambda}_i,\vec{\Lambda}_j) = (H'_S,\infty)$. That is, a sea state of level H'_S is an $[H_S,[H'_S,\infty))$ sea state, in our notation.

4. General framework for long-term stochastic analysis and prediction

From now on we adopt a probabilistic point of view, considering the sea-surface elevation over a multiyear period as a part of a sample function of a non-stationary stochastic process. It should be noted that this randomization is essentially different from the short-term (first-level) randomization, which is previously used to derive the time history of the spectral parameters $\vec{\Lambda}(\tau)$ from a given long-term record $\eta(t)$. Clearly, such a long-term (second-level) randomization of $\eta(t)$ leads to the randomization of the time history of spectral parameters $\vec{\Lambda}(\tau)$, and, as a consequence, of the sequence of sea states $\Xi(i)$. Thus, the following three stochastic processes come now into existence:

- The (continuous-time) primary process "long-term time history of the sea-surface elevation at a given location"

$$\eta(t) = \{\eta(t,\beta), \quad -\infty < t < \infty, \quad \beta \in B\}$$

where β is a choice variable denoting the realizations, and B is an appropriate sample space.

- The (continuous-time) derived process "time history of spectral parameters of the sea states occurring at a given location"

$$\vec{\Lambda}(\tau) = \{\vec{\Lambda}(\tau,\beta), \quad -\infty < \tau < \infty, \quad \beta \in B\}$$

[16]Here and subsequently we use the notation $\Xi(i)$ to denote the whole sequence, in contrast to the notation Ξ_i, which is used to denote the ith element of the sequence.

• The (discrete-time) derived process "sequence of $(\vec{\Lambda},[\vec{\Lambda}_p,\vec{\Lambda}_q))$ sea states occurring at a given location"

$$\Xi(i) = \{\Xi_i(\beta) = (T_{bi}(\beta),T_{ei}(\beta),\vec{\Lambda}_i(\beta)), \quad i = 0,\pm1,\pm2,\ldots, \beta \in B\}$$

Note that the first process is defined on the fast (fine) time scale t, while the second and the third processes are defined on the slow (coarse) time scale τ. (The discrete index i actually counts successive intervals on the τ-time axis.) Since many events related to the primary process (such as the number of peaks above a given level u) become "invisible" when switching from the primary process to the derived processes, the former will be also called the fine process, and the latter the coarse processes.

But why are we concerned with all three processes when two of them can be derived from the other?

The answer to this crucial question is as follows. Actually, we are interested in calculating mean values and probabilities referring to point processes defined on the fine process $\eta(t)$. We know nothing, however, about the statistics of this process, since we cannot assume it ergodic or stationary, and, moreover, we cannot have at our disposal more than one realization of it (usually not even one; see Section 3). It seems thus impossible to directly determine the sought-for quantities by working with the fine process alone. Nevertheless, we should not discard it, since the quantities we are interested in cannot be defined without referring to it. On the other hand, the coarse processes, although inadequate for fine long-term analysis, have the great advantage that their realizations can be derived by measurements or hindcasting techniques, so that their statistics can be determined, under certain assumptions of course, by analyzing existing or readily produced wave data. Under these circumstances, the following approach suggests itself: We shall try to calculate mean values and probabilities of events defined on the fine process, in terms of the statistics of the coarse processes. This is why we retain both the fine and the coarse processes in our analysis. The reasons for retaining both coarse processes are rather technical and less clear for the moment. Note that it seems, in principle, possible to carry out all of our analysis by using only the continuous-time stochastic process $\vec{\Lambda}(\tau)$, but this would require a complete characterization of its stochastic structure. Such an attempt is currently under way, but it is quite complicated since $\vec{\Lambda}(\tau)$ cannot be considered neither as normal nor as stationary.[17] On the other hand, working with the discrete-time stochastic process $\Xi(i)$ is quite natural and simpler.

To proceed rigorously along the lines stated above it is very helpful to construct an inverse of the operator $\mathcal{F}$ defined in Section 3, that is, a procedure $\mathcal{F}^{-1}$ enabling us to associate a time history of the sea-surface elevation $\eta_H(t,\beta)$ to each given time history $\vec{\Lambda}(\tau,\beta)$ of the spectral parameters, or to each given sea-state sequence $\Xi(i,\beta)$. Clearly, two variants of $\mathcal{F}^{-1}$ are possible:

$$\eta_H(t,\beta) = \mathcal{F}_\Lambda^{-1}(\vec{\Lambda}(\tau,\beta)) \qquad \eta_H(t,\beta) = \mathcal{F}_\Xi^{-1}(\Xi(i,\beta))$$

Since, however, there exists a one-to-one correspondence between $\vec{\Lambda}(\tau,\beta)$ and $\Xi(i,\beta)$, the distinction between $\mathcal{F}_\Lambda^{-1}$ and $\mathcal{F}_\Xi^{-1}$ is not essential, and, generally, we shall use the symbol $\mathcal{F}^{-1}$ for representing either $\mathcal{F}_\Lambda^{-1}$ or $\mathcal{F}_\Xi^{-1}$.

Since *a* $\vec{\Lambda}(\tau,\beta)$ is obtained from an $\eta(t,\beta)$ by using some averaging procedure (first-level randomization), it is clear that a complete (strict) inversion of $\mathcal{F}$ is not likely to be possible. However, as will be shown below, it is possible to define a generalized (multiple-valued) right inverse $\mathcal{F}^{-1}$, which generates a substitute $\eta_H(t)$ of the fine process $\eta(t)$, sharing with the latter a lot of statistical information.

Bearing in mind that the time scale of the primary process $\eta(t)$ is faster than the time scale of the derived process $\vec{\Lambda}(\tau)$, we understand that the main step in constructing the operator $\mathcal{F}^{-1}$ will be the realization of an appropriate stretching (dilation) of the slow time τ in order to retrieve the fine time scale t together with the associated phenomena. We shall first proceed to define the key tool for this purpose.

To fix ideas, consider that the spectrum $S(\omega)$ of a sea state is uniquely determined from the vector $\vec{\Lambda} = (\Lambda_1,\Lambda_2, \ldots, \Lambda_L)$. For example, we can take $S(\omega)$ to be the Bretschneider or the modified JONSWAP [Bales et al (1981)] spectrum if $L = 2$, the three-parameter Ochi & Hubble (1976) spectrum if $L = 3$, and so on. However, the choice of a spectral model is not essential, since the final results depend only on the spectral parameters $(\Lambda_1,\Lambda_2, \ldots, \Lambda_L)$. [See the comments after equation (19) in Section 5.] In this sense, each value $\vec{\Lambda}_* = \vec{\Lambda}(\tau_*,\beta_*)$ gives rise to a specific spectrum $S(\omega;\tau_*,\beta_*)$ which, in its turn, generates a specific short-term stochastic process, denoted by

$$\eta^{ST}(t;\tau_*,\beta_*) = \{\eta^{ST}(t,\gamma;\tau_*,\beta_*), \ -\infty < t < \infty, \quad \gamma \in \Gamma(\tau_*,\beta_*)\}$$

Here t is a fast time variable (since it is the dual variable of the spectral frequency ω), which can be locally identified with the time variable of the primary process $\eta(t)$, γ is a choice variable denoting the realizations, and $\Gamma = \Gamma(\tau_*,\beta_*)$ is an appropriate sample space, dependent on the slow-time instant τ_* and the choice variable β_*.

The process $\eta^{ST}(t;\tau_*,\beta_*)$ will be called the "short-term stochastic process generated by $\vec{\Lambda}(\tau_*,\beta_*)$" and it is assumed stationary with respect to the fast time t (t-stationary). Now, considering $\eta^{ST}(t;\tau,\beta)$ for all possible values of τ and β, we obtain a family of short-term stochastic processes generated by the coarse stochastic process $\vec{\Lambda}(\tau)$. This somewhat curious object realizes the sought-for dilation of the slow time, in a way which will be made precise in the next paragraph.

Let $\vec{\Lambda}(\tau,\beta)$ be a given sample function of the continuous-time coarse stochastic process. Let also $\{T_{bi}(\beta)\}$, $\{T_{ei}(\beta)\}$, $\{\Delta T_i(\beta)\}$ and $\{\vec{\Lambda}_i(\beta) = \vec{\Lambda}(\tau_i,\beta)\}$ be the sequences defining the beginning, the end, the duration and the intensity of the successive sea states, in accordance with the definitions of Section 3. To each i (sea state) we associate a part $\eta_i(t,\gamma_i;\beta) = \eta^{ST}(t,\gamma_i;\tau_i,\beta)$, $t \in [T_{bi}(\beta),T_{ei}(\beta)]$, of a sample path γ_i of the short-term stochastic process generated by $\vec{\Lambda}(\tau_i,\beta)$. Note that the fast-time variable t runs throughout the time interval $(T_{bi}(\beta),T_{ei}(\beta))$ which corresponds to the ith sea state. Now we define the operator $\mathcal{F}^{-1}$ as follows

$$\eta_H(t,\beta) = \begin{cases} \mathcal{F}^{-1}(\Lambda(\tau,\beta)) \\ \quad\text{or} \\ \mathcal{F}^{-1}(\Xi(i,\beta)) \end{cases} = \begin{cases} \ldots\ldots\ldots\ldots \\ \eta_{-i}(t,\gamma_{-i};\beta), \text{ for } T_{b,-i}(\beta) < t < T_{e,-i}(\beta) \\ \ldots\ldots\ldots\ldots \\ \eta_0(t,\gamma_0;\beta), \text{ for } T_{b0}(\beta) < t < T_{e0}(\beta) \\ \ldots\ldots\ldots\ldots \\ \eta_i(t,\gamma_i;\beta), \text{ for } T_{bi}(\beta) < t < T_{ei}(\beta) \\ \ldots\ldots\ldots\ldots \end{cases} \quad (9)$$

[17] The stochastic structure under examination is the one of a periodically correlated process with log-normal densities.

This stochastic process $\{\eta_H(t,\beta), -\infty < t < \infty, \beta \in B\}$ will be called "the hindcasted[18] long-term sea-surface elevation."

In order to describe informally the above definition it is very helpful to borrow the idea of "time windows," introduced by Burridge et al (1987) in a paper dealing with the propagation of acoustic waves through a slightly nonhomogeneous random medium. Using this notion we can describe our definition as follows. To each τ_i (a reference time instant of the ith sea state), we associate a time window of duration $\Delta T_i(\beta)$ within which $\eta^{ST}(t,\gamma_i;\tau_i,\beta)$ represents the sea-surface elevation, with t measuring the fast time. A complete long-term sea-surface elevation sample function is then taken by putting together the parts $\eta^{ST}(t,\gamma_i;\tau_i,\beta)$, appearing in successive time windows.

On the other hand, the above definition of $\eta_H(t,\beta)$ strongly reminds the definition of a regenerative stochastic process [see for example Smith (1955,1958), Section 2.1, or Klimov (1986), Chapter 5)], the pair $((T_{bi},T_{ei}),\eta_i(t,\gamma_i;\beta))$ being a tour or a cycle of duration $\Delta T_i = T_{ei} - T_{bi}$. Note, however, that in the classical theory of a sequence of cycles $C_i = (Z_i,f_i(t,\gamma_i))$ forms a regenerative process if

1. all C_i are identically distributed;
2. any pairs C_i, C_j, $i \neq j$, are independently distributed; or
3. all sample-path functions $f(t,\gamma_i)$ belong to the same sample space, that is, $\gamma_i \in \Gamma$, Γ being independent of i.

As regards the process $\eta_H(t,\beta)$, in which we are interested in the present work, it does not satisfy all the above assumptions. Actually only assumption 1 seems indispensable for what follows. Assumption 2 will be introduced in the next section for the sake of simplicity, but the construction of a more general model with some kind of dependence between the successive cycles seems to be both possible and useful [Labeyrie (1990)]. Finally, assumption 3 is by no means applicable in our case, where the sample-path segments $\eta_i(t,\gamma_i;\beta)$ belong to a population of sample spaces $\Gamma(\beta)$ whose defining parameter β ranges over another sample space. This fact characterizes our model as a two-level stochastic process. Note, however, that we do not intend to study in depth such a stochastic process in the context of the present paper.

Let us comment a little more on the definition (9). Clearly, $\mathcal{F}^{-1}$ is not single-valued, since the sample paths γ_i, $i = 0,1,2,\ldots$, of the short-term process appearing in the right-hand side of equation (9) are quite arbitrary. In this sense it is enlightening to denote the "hindcasted" sample function of the fine process by $\eta_H(t,\beta;\{\gamma_i\})$. Nevertheless, the operator $\mathcal{F}^{-1}$ is a generalized right inverse of the operator $\mathcal{F}$ in the sense that

$$\mathcal{F}(\mathcal{F}^{-1}(\vec{\Lambda}(\tau,\beta)) = \mathcal{F}(\eta_H(t,\beta;\{\gamma_i\}) = \vec{\Lambda}(\tau,\beta) \qquad (10)$$

by the very definition of $\eta_H(t,\beta;\{\gamma_i\})$.[19] On the other hand, it is clear that

$$\mathcal{F}^{-1}(\mathcal{F}(\eta(t,\beta)) = \eta_H(t,\beta;\{\gamma_i\}) \neq \eta(t,\beta) \qquad (11)$$

Having the operator $\mathcal{F}^{-1}$ at our disposal, we can legitimately expect that various statistical characteristics of the fine process $\eta(t)$ would be inferred from the statistics of the coarse processes $\vec{\Lambda}(\tau)$ or $\Xi(i)$, by means of an appropriate theoretical analysis. An example of such an analysis will be given in the next section, where the long-term mean number of peaks above a given level u will be calculated. Before proceeding towards this direction, it is advisable to briefly discuss some questions concerning the statistical characterization of the process $\Xi(i)$. We shall restrict our attention to the first-order statistics of the process, that is, to the pdfs $f_i(\Delta T,\vec{\Lambda}) = f(\Delta T_i,\vec{\Lambda}_i)$, since this is the only statistical information needed in the applications taken up in this paper.

Clearly, some assumptions are necessary in order to make possible a practically realizable method to obtain empirical approximants of the needed pdfs. The fundamental assumption made in this work is that $f_i(\Delta T,\vec{\Lambda})$ is independent of the order i of the sea state. This is a first-order stationarity hypothesis for the coarse process $\Xi(i)$, which will be referred to as the long-term (second-level) stationarity, in order to be distinguished from the short-term (first-level) stationarity of the process $\eta(t)$ in short-term periods. Because of seasonal effects, even this long-term stationarity is questionable (at least for some kinds of applications), if $\Xi(i)$ is constructed by processing entire (multiyear) continuous records $\vec{\Lambda}(\tau,\beta)$. (Seasonal and cyclic trends are expected to be present in such records $\vec{\Lambda}(\tau,\beta)$). To circumvent this difficulty, we can subdivide the annual parts of each record $\vec{\Lambda}(\tau,\beta)$ into seasonal parts, and obtain a number of long-term seasonal records $\vec{\Lambda}_x(\tau,\beta)$, $\vec{\Lambda}_\psi(\tau,\beta)$, ... by putting together the various parts corresponding to the same season. (The word "season" is used here with the meaning explained in Section 1, footnote 9.) Then, each of the derived (discrete) processes $\Xi_x(i)$, $\Xi_\psi(i)$, ..., can be considered stationary [Battjes (1977), Section 4.1] and this assumption may be expressed as seasonal stationarity. In the sequel we shall discard the subscripts X,Ψ, ..., considering $\vec{\Lambda}(\tau)$ and $\Xi(i)$ as seasonal. With this reserve we shall proceed assuming that $f_i(\Delta T,\vec{\Lambda}) = f(\Delta T,\vec{\Lambda})$, the latter being independent of i. This is the essence of our long-term stationarity assumption, which is in fact equivalent to the assumption 1 above.

To obtain an empirical approximant of the pdf $f(\Delta T,\vec{\Lambda})$, we have to make a careful multivariate frequency analysis, using a sufficiently long (multiyear) record $\vec{\Lambda}(\tau,\beta)$. In this sense, a long-term (second-level) ergodicity hypothesis should be also made.

At this point it is appropriate to list the various pdfs related to the first-order statistics of the coarse processes, and to briefly comment on the relations between them. The basic first-order densities are:

- the above described joint pdf $f(\Delta T,\vec{\Lambda})$;
- the marginal pdfs $f_{mg}(\Delta T)$ and $f_{mg}(\vec{\Lambda})$, obtained by integrating $f(\Delta T,\vec{\Lambda})$;
- the usual first-order pdf of the stationary stochastic process $\vec{\Lambda}(\tau)$, which will be denoted by $f_{cl}(\vec{\Lambda})$, the subscript cl coming from the word classical; and
- the conventional scatter diagram $f_{sd}(\vec{\Lambda})$, where $\vec{\Lambda} = (H_S,T_0)$.

In general, $f_{mg}(\vec{\Lambda}) \neq f_{cl}(\vec{\Lambda}) \neq f_{sd}(\vec{\Lambda})$, while $f(\Delta T,\vec{\Lambda})$ is related to $f_{cl}(\vec{\Lambda})$ by a rather surprising relation, which will be proved in Section 7. However, $f_{sd}(\vec{\Lambda})$ can be, with some precautions, considered as an empirical approximant of $f_{cl}(\vec{\Lambda})$.

5. Probability distribution of the long-term number of peaks above a given level

The long-term stochastic framework constructed in the previous sections can be used to study a number of interesting problems. For the sake of definiteness, and because of its great importance in practical applications, we shall, in this section, focus our attention on the specific event (point process): "the number of peaks (local maxima) of the primary process $\{\eta(t,\beta),\beta \in B\}$ lying above a level u and occurring in a (long-term) time interval $[T_1,T_2]$." This quantity will be denoted by $M_u([T_1,T_2];\beta)$ or $M_u(T_1,T_2,\beta)$.

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[18] Here, the term "hindcasted" is used to describe a theoretical procedure and not the result of a numerical simulation procedure.

[19] The second equality in the relation (10) is, in fact, approximate, but this does not affect the reasoning.

As the notation clearly indicates, the quantity $M_u([T_1,T_2];\beta)$ depends on two variables: the choice variable β, denoting the sample function on which the number of peaks is counted, and the time interval $[T_1,T_2]$ during which the counting process takes place. Note that any well-defined subset S of the time axis, for example, a union of intervals, can be used in place of $[T_1,T_2]$, in which case we shall use the notation $M_u(S;\beta)$. Perhaps the most fundamental property of $M_u(S;\beta)$ is its additivity with respect to the set variable S, that is

$$M_u(S_1 \cup S_2,\beta) = M_u(S_1;\beta) + M_u(S_2;\beta) \qquad (12)$$

whenever $S_1 \cap S_2 = \phi$. Here the symbols $\cup$, $\cap$, and ϕ denote the set-theoretic union, the set-theoretic intersection and the null set, respectively. The above two properties of $M_u(S;\beta)$, namely its β-randomness and its S-additivity, characterize it as a random measure. Clearly, when S is fixed, $M_u(S;\beta)$ is reduced to an ordinary random variable. Let it be noted here that all the analysis performed in this section applies unchanged to a large class of very important random measures defined on the primary process $\eta(t)$. See Appendix 1, where this class is defined and various examples of random measures (point processes) belonging to it are given.

Let T be a fixed long-term period of interest, for example, 20 or 50 years (or winters). Without loss of generality we can assume that this time period covers the interval $[0,T]$. We are especially interested in the random variable $M_u([0,T];\beta)$, for which we shall use the shorthand notation $M_u(T;\beta)$. Our purpose in this section is to determine the complete probability structure of this quantity in terms of the statistics of the coarse process $\Xi(i)$. As far as we know, only the problem of determining the mean value $\mathbf{E}^{\beta}[M_u(T;\beta)]$[20] of the quantity $M_u(T;\beta)$ has been studied up to now [Battjes (1970)].

Before proceeding it is necessary to define one more random quantity, namely the number of sea states in the time interval $[0,T]$, denoted by $N(T;\beta)$. Actually, this is also a random measure, defined on the continuous-time coarse process $\vec{\Lambda}(\tau)$. If, however, we assume that T is fixed and we disregard portions of sea states falling at the ends of the interval $[0,T]$, then $N(T,\beta)$ becomes an integer-valued random variable. Clearly

$$T = \sum_{i=1}^{N(T;\beta)} \Delta T_i(\beta) + T_{b1} + (T - T_{eN}) \qquad (13)$$

where $\Delta T_i(\beta) = T_{ei}(\beta) - T_{bi}(\beta)$, and $T_{bi}(\beta)$ and $T_{ei}(\beta)$ denote the beginning and the end of the ith sea state, respectively. The first and the Nth sea state are, by definition, the first sea state starting after $\tau = 0$ and the last sea state ending before $\tau = T$, respectively. Note that T_{bi} and $T - T_{eN}$ are of the order $0(\hat{T}_2)$, while $T = 0(\hat{T}_3)$ (see Section 2). Thus, according to the order assumption (7c), we have

$$\frac{T_{b1} + (T - T_{eN})}{T} << 1 \qquad (14)$$

which permits us to disregard the last two terms in the right-hand side of equation (13).

Let us now return to the study of the random quantity $M_u(T;\beta)$. In virtue of its additivity, we have

$$M_u(T;\beta) = \sum_{i=1}^{N(T;\beta)} M_u(T_{b1}(\beta),T_{e1}(\beta);\beta) + M_u(0,T_{b1};\beta) + M_u(T_{eN},T;\beta) \qquad (15)$$

[20]$\mathbf{E}^{\beta}[\ldots]$ is the ensemble average operator extended over the sample space B. The use of the choice variable β as a superscript is crucial, since we shall subsequently use ensemble average operators extended over different sample spaces, as well as repeated ensemble averages; see, for example, equation (17b) below.

Normally, we may expect that

$$\frac{M_u(0,T_{b1};\beta) + M_u(T_{eN},T;\beta)}{M_u(T;\beta)} = 0(\hat{T}_2/\hat{T}_3) << 1 \qquad (16)$$

which permits us to disregard the last two terms in the right-hand side of equation (15).

At this point we shall introduce a key assumption, permitting us to efficiently blend the short- and the long-term levels. Let us first define the quantity $M_u^{ST}(T_{bi}(\beta),T_{ei}(\beta);\gamma_i)$, representing the number of peaks lying above the level u and occurring during the part $[T_{bi}(\beta),T_{ei}(\beta)]$ of the short-term sample function γ_i, corresponding to the ith sea state. [See Section 4, equation (9)]. Then, our key assumption reads as follows.

Hierarchy assumption: The number $M_u(T_{bi}(\beta),T_{ei}(\beta);\beta)$ of peaks lying above u and occurring during the ith sea state can be approximated by the number $\mathbf{E}^{\gamma_i}[M_u^{ST}(T_{bi}(\beta),T_{ei}(\beta);\gamma_i)]$, that is, the mean value of $M_u^{ST}(T_{bi}(\beta),T_{ei}(\beta);\gamma_i)$ over all short-term sample functions γ_i. In symbols:

$$M_u(T_{bi}(\beta),T_{ei}(\beta);\beta) \cong \mathbf{E}^{\gamma_i}[M_u^{ST}(T_{bi}(\beta),T_{ei}(\beta);\gamma_i)] \qquad (17a)$$

We call this assumption the hierarchy assumption, since it permits us to express the long-term mean value of $M_u(T;\beta)$ by means of a two-level hierarchical procedure:

$$\mathbf{E}^{\beta}[M_u(T;\beta)] \cong \mathbf{E}^{\beta}\left[\sum_{i=1}^{N(T;\beta)} \mathbf{E}^{\gamma_i}[M_u^{ST}(T_{bi}(\beta),T_{ei}(\beta);\gamma_i)]\right] \qquad (17b)$$

As regards the validity of this assumption, we feel that it is quite plausible, at least under the order assumption (7a). In any case, it can be subjected to direct experimental verification, since the left-hand side of equation (17a) can be easily measured on a continuous record of the sea-surface elevation, while its right-hand side can be calculated in terms of the spectral characteristics of the corresponding sea states by means of the formula

$$\begin{aligned}\mathbf{E}^{\gamma_i}[M_u^{ST}(T_{bi}(\beta),T_{ei}(\beta);\gamma_i)] &= M(u;\Delta T_i(\beta),\vec{\Lambda}_i(\beta)) \\ &= \Delta T_i(\beta) M(u;1,\vec{\Lambda}_i(\beta)) \\ &= \Delta T_i(\beta) W(u;\vec{\Lambda}_i(\beta)) \\ &\equiv \Delta T_i(\beta) W_i(\beta) \equiv M_i(\beta) \qquad (18)\end{aligned}$$

where $\vec{\Lambda}_i(\beta) = (m_{0i}(\beta),m_{2i}(\beta),m_{4i}(\beta))$, m_{ki} is the kth spectral moment of the ith sea state, and $W(u;\vec{\Lambda}_i(\beta)) = M(u;1,\vec{\Lambda}_i(\beta))$ is the mean number of peaks per unit (fast) time occurring in a sea state with intensity $\vec{\Lambda}_i(\beta)$. The quantity $W(u;\vec{\Lambda})$ is dependent only on the short-term stochastic model and can be considered known. Especially for a Gaussian short-term process, this quantity has been first calculated by Rice (1954), and is repeated in Appendix 2, equation (46), for easy reference. Now using equations (18) and exploiting all the assumptions described above, we can write equation (15) in the form

$$M_u(T;\beta) = \sum_{i=1}^{N(T;\beta)} M_i(\beta) = \sum_{i=1}^{N(T;\beta)} \Delta T_i(\beta) W_i(\beta) \qquad (19)$$

Equation (19) determines, in conjunction with equations (18), which characteristics of the individual sea states should be taken into account when calculating the probability structure of the random quantity $M_u(T;\beta)$. Under the assumption of normality for the short-term process, these characteristics are the spectral moments[21] m_0,m_2,m_4; see Ap-

[21]Or, equivalently, the spectral characteristics σ (or H_S), T_0, and ε, where σ is the standard deviation of the sea-surface elevation, and ε is the broadness coefficient of the corresponding short-term spectrum.

pendix 2 equations (43)–(46). On the other hand, equation (19) expresses $M_u(T;\beta)$ as a random sum of identically distributed random variables. For all $M_i(\beta)$, $i = 1,2, \ldots$, are identically distributed since each $M_i(\beta)$ is a deterministic function of the random vector $(\Delta T_i(\beta),\vec{\Lambda}_i(\beta))$ and, according to the long-term seasonal stationarity assumption, all $\Delta T_i(\beta)$, $i = 1,2, \ldots$, and all $\vec{\Lambda}_i(\beta)$, $i = 1,2, \ldots$, are identically distributed.

In general, the random vectors $(\Delta T_i(\beta),\vec{\Lambda}_i(\beta))$, $i = 1,2, \ldots$, are expected to be dependent. For example, it seems very likely that the appearance of a specific value $(\Delta T_k(\beta),\vec{\Lambda}_k(\beta))$ for the duration and intensity of a sea state would imply some restrictions on the corresponding values of the next sea state [Laviel & Rio (1987)]. However, in the present work we shall adopt the following

Independence assumption: The random vectors $(\Delta T_i(\beta),\vec{\Lambda}_i(\beta))$, $i = 1,2, \ldots$, and consequently the random variables $M_i(\beta)$, $i = 1,2, \ldots$, are statistically independent of each other.

Under these assumptions all $\Delta T_i(\beta)$, $i = 1,2, \ldots$, and all $M_i(\beta)$, $i = 1,2, \ldots$, become independently and identically distributed (i.i.d.) random variables, and then equation (19) lends the structure of a renewal-reward (cumulative) process to the random quantity $M_u(T;\beta)$. More precisely:

1. The time instants $T_{bi}(\beta)$, (or $T_{ei}(\beta)$), $i = 1,2, \ldots$, define a renewal point process whose successive interarrival times are the durations $\Delta T_i(\beta)$ of successive sea states. The renewal function of this process is $\mathbf{E}^{\beta}[N(T;\beta)]$, that is, the mean number of sea states occurring in the (long-term) period $[0,T]$.
2. The sequence $\{(\Delta T_i(\beta),M_i(\beta)),\ i = 1,2, \ldots,\}$ defines a renewal-reward (cumulative) process in which $M_i(\beta)$ is the "cost" (or the "reward") associated with the ith interarrival time $\Delta T_i(\beta)$, and $M_u(T;\beta)$ is the up-to-time T "accumulated cost" (or the "total reward") whose probability structure is our main concern in this section.

Under these circumstances, the arsenal of the renewal (and renewal-reward) theory [Smith (1958), Cox (1964), Ross (1970), Brown & Ross (1972), Karlin & Taylor (1975)] becomes available, making the determination of the probability structure of the quantities $N(T;\beta)$ and $M_u(T;\beta)$ a simple yet useful exercise. Especially, we can obtain explicit asymptotic (for $T \to \infty$) expressions for the mean values, the variances, and the joint pdf of $N(T;\beta)$ and $M_u(T;\beta)$.

Before presenting the aforementioned results, and to keep their appearance as simple as possible, we have to introduce some terminology. Let us first agree to discard the subscript i (order of the sea state) and the choice variable β, since from now on all random variables of the form X_i, $i = 1,2, \ldots$, are i.i.d., and they are all defined on the same sample space B. Define now the auxiliary "density" functions[22]

$$g_m(\vec{\Lambda}) = \frac{1}{\mu_{m,\Delta T}} \int_0^{\infty} (\Delta T)^m f(\Delta T,\vec{\Lambda})\, d(\Delta T),\ m = 1,2, \ldots, \tag{20}$$

where

$$\mu_{m,\Delta T} = \mathbf{E}[(\Delta T)^m] = \int_{\vec{\Lambda}} \int_0^{\infty} (\Delta T)^m f(\Delta T,\vec{\Lambda})\, d(\Delta T) d\vec{\Lambda} \tag{21}$$

is the mth-order moment of the random variable ΔT. In equation (21) $d\vec{\Lambda} = dm_0 dm_2 dm_4$ and the $\vec{\Lambda}$-integration extends over the corresponding three-dimensional region. Consider also the moments $\mu_{m,M}$ of the random variable $M \equiv M(u;\Delta T,\vec{\Lambda})$, which, with the aid of equations (18) and (20), can be written as

[22]All $g_m(\vec{\Lambda})$, $m = 1,2, \ldots$, are defined so that their $\vec{\Lambda}$-integral is equal to unity. Besides, in Section 7, we shall prove that $g_1(\vec{\Lambda})$is indeed a pdf.

$$\begin{aligned}\mu_{m,M} &= \mathbf{E}[M(u;\Delta T,\vec{\Lambda})^m] \\ &= \mathbf{E}[(\Delta T)^m (W(u;\vec{\Lambda}))^m] \\ &= \mu_{m,\Delta T} \int_{\vec{\Lambda}} (W(u;\vec{\Lambda}))^m g_m(\vec{\Lambda})\, d\vec{\Lambda},\ m = 1,2, \ldots\end{aligned} \tag{22}$$

and the variances of ΔT and M, given by

$$\sigma^2_{\Delta T} \equiv \mathrm{Var}[\Delta T] = \mu_{2,\Delta T} - \mu^2_{\Delta T} \tag{23}$$

$$\sigma^2_M \equiv \mathrm{Var}[M] = \mu_{2,M} - \mu^2_M \tag{24}$$

where

$$\mu_{\Delta T} \equiv \mu_{1,\Delta T} \quad \text{and} \quad \mu_M \equiv \mu_{1,M} \tag{25}$$

are simplified notations for the mean values of ΔT and M, respectively. Finally, consider the correlation coefficient of ΔT and M, given by

$$\rho_{\Delta T,M} \frac{\mathbf{E}[(\Delta T)M] - \mu_{\Delta T}\mu_M}{\sigma_{\Delta T}\, \sigma_M} \tag{26}$$

where

$$\begin{aligned}\mathbf{E}[(\Delta T)M] &\equiv \mathbf{E}[(\Delta T)^2 W(u;\vec{\Lambda})] \\ &= \mu_{2,\Delta T} \int_{\vec{\Lambda}} W(u;\vec{\Lambda})\, g_2(\vec{\Lambda})\, d\vec{\Lambda}\end{aligned} \tag{27}$$

The quantities $\mu_{\Delta T},\mu_M,\sigma_{\Delta T},\sigma_M$, and $\rho_{\Delta T,M}$ defined above constitute the fundamental probabilistic characteristics of the bivariate random quantity $(\Delta T,M)$, where $M = M(u;\Delta T,\vec{\Lambda})$. From their definitions, equations (21)–(26), it can be easily seen that all these quantities are expressed in terms of the functions $f(\Delta T,\vec{\Lambda})$ and $W(u;\vec{\Lambda})$, the former describing the statistics of the sea-state population associated with the examined ocean site, and the latter expressing the short-term mean value of the studied random measure for a given sea state. Thus, the five quantities $\mu_{\Delta T},\mu_M,\sigma_{\Delta T},\sigma_M$, and $\rho_{\Delta T,M}$ can be considered known.

Now, by using standard results from the theory of renewal-reward (cumulative) processes [Smith (1958), Section 2.3; Cox (1962), Chapter 8; Brown & Ross (1972)], we obtain the following theorem:

Theorem: For large values of T, the joint pdf of $N(T)$ and $M_u(T)$ is a bivariate normal distribution whose parameters are given by the equations

$$\mathbf{E}[N(T)] = \frac{T}{\mu_{\Delta T}} + 0(1) \tag{28}$$

$$\mathbf{VAR}[N(T)] = \frac{T}{\mu_{\Delta T}} \frac{\sigma^2_{\Delta T}}{\mu^2_{\Delta T}} + 0(1) \tag{29}$$

$$\mathbf{E}[M_u(T)] = \frac{T}{\mu_{\Delta T}} \mu_M + 0(1) \tag{30}$$

$$\begin{aligned}\mathbf{VAR}[M_u(T)] = \frac{T}{\mu_{\Delta T}} (\sigma^2_M &+ \sigma^2_{\Delta T}\mu^2_M/\mu^2_{\Delta T} \\ &- 2\rho_{\Delta T,M}\sigma_{\Delta T}\sigma_M\, \mu_M/\mu_{\Delta T}) + 0(1)\end{aligned} \tag{31}$$

$$\begin{aligned}&\mathbf{Cov}[N(T),M_u(T)] \\ &\quad = \frac{T}{\mu_{\Delta T}} \frac{\sigma^2_{\Delta T}}{\mu^2_{\Delta T}} (\mu_M - \mu_{\Delta T}\rho_{\Delta T,M}\sigma_M/\sigma_{\Delta T}) + 0(1)\end{aligned} \tag{32}$$

Thus, by using the renewal-reward theory, we manage to obtain the complete probability structure of the quantities $N(T)$ and $M_u(T)$. Note that the above results can be safely used for actual long-term calculations, since $T/\mu_{\Delta T}$ is of the order of $T_3/T_2 >> 1$; see the order assumption (7c).

Let us now examine more closely the result (30), that is, the mean number of peaks above a level u occurring in a long-term time period $[0,T]$. Combining equations (20), (22), (30), and (46) of Appendix 2, we obtain

$$\mathbf{E}[M_u(T)] = T \int_{\vec{\Lambda}} W(u;\vec{\Lambda}) g_1(\vec{\Lambda})\, d\vec{\Lambda} = T \int_{\vec{\Lambda}} \frac{1 - G(u;\sigma,\varepsilon)}{\delta\, T_0} g_1(\sigma,T_0,\varepsilon) d\vec{\Lambda} \qquad (33)$$

where $d\vec{\Lambda} = d\sigma dT_0 d\varepsilon$. If we make the additional assumption that $G(u;\sigma,\varepsilon)$ is ε-independent[23] and use the change of variable $H_S = 4\sigma$, we obtain

$$\mathbf{E}[M_u(T)] = T \int_0^\infty \int_0^\infty \frac{\exp(-8u^2/H_S^2)}{T_0} g_1(H_S,T_0) dH_S dT_0 \qquad (34)$$

where $g_1(H_S,T_0)$ is the marginal "density" taken from $g_1(\sigma,T_0,\varepsilon)$ by integrating with respect to ε, and using the change of variable $H_S = 4\sigma$.

Equation (34) is formally identical with the corresponding result of Battjes (1970), with $g_1(H_S,T_0)$ in place of the scatter diagram $f_{sd}(H_S,T_0)$. However, in Section 7 we shall prove that $g_1(H_S,T_0) = f_{cl}(H_S,T_0)$. Accordingly, whenever $f_{sd}(H_S,T_0)$ can be considered as a reliable approximant of $f_{cl}(H_S,T_0)$, then equation (34) becomes essentially identical with the corresponding one obtained by Battjes. See the pertinent comments at the end of Section 7.

Apart from equation (34), all above results are apparently new. Moreover, the whole conceptual framework constructed above permits us to identify and "locate" the various simplifying assumptions which are made in long-term analysis, and makes it possible to construct more realistic models by removing the unfit ones.

In concluding this section we emphasize once again that the above analysis can be literally repeated if we replace $M_u(T;\beta)$ by any other quantity $\chi(T;\beta)$, satisfying the conditions stated in Appendix 1. It is clear that the set $\vec{\Lambda}$ of the spectral parameters involved is generally dependent on the specific quantity considered. This suggests the necessity of having available the joint pdf $f(\Delta T,\vec{\Lambda})$ for various combinations of spectral parameters $\vec{\Lambda}$. Such data sets do not apparently exist at the moment, but it is feasible to derive them from time histories of spectral density functions obtained with the aid of measurements or hindcasting techniques. In this sense our theory gives indications concerning new directions of wave data processing and presentation. Note, however, that a definite assessment of the practical significance of our results cannot be made until a reliable estimate for the pdf $f(\Delta T,H_S,T_0)$ becomes available.

6. Long-term probability distribution of the wave amplitude

In this section we shall rigorously define and calculate the long-term probability distribution of the zero-to-crest wave amplitude a. The long-term probability distribution of the crest-to-trough wave height H will be given only under the assumption of narrow-band sea states. A brief discussion comes first in order to make clear the essential conceptual difference between the short-term and the long-term probability distributions of the wave amplitude, which is usually masked by the similarity of the final formal definitions.

Let us first recall the definition of a peak in a continuous-time stochastic process. A peak (local maximum) occurs at $t = t_0$ in a realization $\eta(t,\beta)$ of a (differentiable) stochastic process, whenever the first derivative $d\eta(t,\beta)/dt$ has a down-crossing of zero at $t = t_0$. Clearly, this definition is unambiguous and well-grounded for any kind of stochastic process, stationary or non-stationary, ergodic or non-ergodic. Unfortunately, the same is not the case for the definition of the probability distribution $P(a) = \mathbf{P_R}[\text{wave amplitude} \le a]$. In fact, the standard definition of $P(a)$ encountered in electrical, structural and ocean engineering literature [Price & Bishop (1974), Ochi (1982), Middletton (1960), Rice (1954), Lin (1967)], namely

$$P(a) = \frac{\text{Mean value of peaks per unit time below } a}{\text{Mean value of all peaks per unit time}} \qquad (35)$$

is unambiguous and can be rigorously justified only for (strictly) stationary processes [Cramer & Leadbetter (1967), Section 11.6]. Accordingly, this definition can be used only in the short-term (stationary) case. For non-stationary processes the above definition is clearly meaningless, while the following rigorous one [Cramer & Leadbetter (1967)]

$$P(a) = \lim_{\theta \to 0} \mathbf{P_R}[\eta(t_0) \le a \mid \eta'(t) \text{ has a downcrossing of zero in } [t_0 - \theta, t_0]]$$

leads to a probability distribution $P(a)$ dependent on t_0.

The above observations clearly show that a carefully designed definition of the probability distribution of the wave amplitude a is needed in our case, where the process under consideration is stationary in the short-term scale and non-stationary in the long-term one. Bearing in mind that the mean value of peaks of $\eta(t)$ over long-term time intervals $[T_1,T_2]$ is proportional to the time length $\Delta T^{LT} = T_2 - T_1$ [see equation (30) or (33)], we can define the long-term probability distribution $P_L(a)$ as follows

$$P_L(a) = \frac{\text{Mean value of peaks below } a \text{ occurring in a long-term period } \Delta T^{LT}}{\text{Mean value of all peaks occurring in } \Delta T^{LT}} \qquad (36)$$

Observe that, if we divide the numerator and the denominator of the right-hand side of equation (36) by ΔT^{LT}, we obtain a relation similar to (35) in which the expression "per unit time" has been replaced by the expression "per unit of the slow time τ." This shows how the distinction of the two time scales can help us in clarifying and properly stating the fundamental definitions.

Now, combining (33) and (36), we obtain the following formula expressing $P_L(a)$ in terms of $W(a,\vec{\Lambda})$ and $g_1(\vec{\Lambda}) = g_1(\sigma,T_0,\varepsilon)$:

$$P_L(a) = \frac{\int_{\vec{\Lambda}} [W(-\infty,\vec{\Lambda}) - W(a,\vec{\Lambda})]\, g_1(\vec{\Lambda})\, d\vec{\Lambda}}{\int_{\vec{\Lambda}} W(-\infty,\vec{\Lambda})\, g_1(\vec{\Lambda})\, d\vec{\Lambda}} \qquad (37)$$

Let us now proceed to examine some special cases. Assuming that ε is a deterministic constant and using (37) and (46) of Appendix 2, we obtain

$$P_L(a) = \frac{\int_0^\infty \int_0^\infty \frac{G(a;\sigma,\varepsilon)}{T_0} g_1(\sigma,T_0) d\sigma dT_0}{\int_0^\infty \frac{1}{T_0} g_1(T_0) dT_0} \qquad (38)$$

[23]Taking for example $G(u;\sigma,\varepsilon) \cong G(u;\sigma,0)$, which is valid for narrow-band sea states.

where $g_1(\sigma,T_0)$ and $g_1(T_0)$ are the corresponding marginal "densities" obtained from $g_1(\sigma,T_0,\varepsilon)$ by integration. Furthermore, assuming that $\varepsilon = 0$, and changing to $H_S = 4\sigma$ and $H = 2a$, we get the corresponding formula for the long-term probability distribution $P_L(H)$ of the crest-to-trough wave height H. This is similar with formula (38) with the term

$$\int_0^\infty \int_0^\infty \frac{R(H;H_S)}{T_0} g_1(H_S,T_0)\, dH_S dT_0$$

as numerator and with the same denominator, where

$$R(H;H_S) = 1 - \exp[-2(H/H_S)^2]$$

is the standard Rayleigh CDF. This result for $P_L(H)$ is the same with Battjes's result [see equation (5)] as long as $g_1(H_S,T_0)$ can be replaced by $f_{sd}(H_S,T_0)$. See the pertinent comments at the end of Section 7.

In ocean engineering, it is sometimes preferable to consider the conditional CDF $P_L^+(a) = \mathbf{P}\textsc{r}$[wave amplitude $\leq a$|the amplitude is non-negative] in which only the positive peaks are taken into account. It is not difficult to find that the general expression for this CDF has a form similar to (37), with $W(0,\vec{\Lambda})$ in place of $W(-\infty,\vec{\Lambda})$.

7. Relation between densities $f(\Delta T,\vec{\Lambda})$ and $f_{cl}(\vec{\Lambda})$

We shall now prove the following relation between the densities $f(\Delta T,\vec{\Lambda})$ and $f_{cl}(\vec{\Lambda})$

$$f_{cl}(\vec{\Lambda}) = \frac{1}{\mu_{\Delta T}} \int_0^\infty \Delta T\, f(\Delta T,\vec{\Lambda})\, d(\Delta T) \equiv g_1(\vec{\Lambda}) \qquad (39)$$

The underlying reasoning in proving this relation is essentially similar to that lying behind the so-called inspection paradox of classical renewal theory [Heyman & Sobel (1982), Section 5.1]. The proof is free of any technical burden; it is of purely conceptual character. Accordingly, it is expedient to start by stating the exact definitions of the two densities:

$$f(\Delta T,\vec{\Lambda})d\vec{\Lambda}d(\Delta T) = \mathbf{P}\textsc{r}\left[\begin{array}{l}\text{duration } \Delta T \text{ and intensity } \vec{\Lambda} \text{ of}\\ \text{a sea state lie in intervals}\\ [\Delta T,\Delta T + d(\Delta T)) \text{ and } [\vec{\Lambda},\vec{\Lambda} + d\vec{\Lambda}),\\ \text{respectively}\end{array}\right]$$

$$f_{cl}(\vec{\Lambda})d\vec{\Lambda} \equiv \pi(\vec{\Lambda},d\vec{\Lambda}) = \mathbf{P}\textsc{r}\left[\begin{array}{l}\text{intensity } \vec{\Lambda} \text{ of a randomly}\\ \text{sampled sea state lies in}\\ \text{interval } [\vec{\Lambda},\vec{\Lambda} + d\vec{\Lambda})\end{array}\right]$$

Define also the conditional probability

$$\pi(\vec{\Lambda},d\vec{\Lambda}|\Delta T_i) = \mathbf{P}\textsc{r}$$

$$\left[\begin{array}{l}\text{intensity } \vec{\Lambda} \text{ of a randomly sampled}\\ \text{sea state lies in interval } (\vec{\Lambda},\vec{\Lambda} + d\vec{\Lambda}),\\ \text{given that its duration } \Delta T_i \text{ lies in interval}\\ (\Delta T,\Delta T + d(\Delta T))\end{array}\right]$$

To find a relation between $f(\Delta T,\vec{\Lambda})$ and $f_{cl}(\vec{\Lambda})$ we think as follows:

Using the total probability formula we obtain

$$\pi(\vec{\Lambda},d\vec{\Lambda}) = \sum_{\Delta T_i} \pi(\vec{\Lambda},d\vec{\Lambda}|\Delta T_i) \qquad (40)$$

However, the probability $\pi(\vec{\Lambda},d\vec{\Lambda}|\Delta T_i)$ can be expressed in terms of the density $f(\Delta T,\vec{\Lambda})$. For the probability $\pi(\vec{\Lambda},d\vec{\Lambda}|\Delta T_i)$ should be proportional to $f(\Delta T_i,\vec{\Lambda})d\vec{\Lambda}$, while it must be also proportional to ΔT_i itself. The latter assertion is implied by the fact that, sampling at random, it is twice as likely to choose a sea state of duration $2\Delta T_i$ as one of duration ΔT_i. Thus

$$\pi(\vec{\Lambda},d\vec{\Lambda}|\Delta T_i) = A\, \Delta T_i f(\Delta T_i,\vec{\Lambda})d\vec{\Lambda} \qquad (41)$$

where A is the coefficient of proportionality which can be assumed independent of $\vec{\Lambda}$. Combining now (40) and (41) we obtain

$$\begin{aligned} f_{cl}(\vec{\Lambda})d\vec{\Lambda} &= A \sum_{\Delta T_i} \Delta T_i f(\Delta T_i,\vec{\Lambda})d\vec{\Lambda} \\ &= A \int_0^\infty (\Delta T) f(\Delta T,\vec{\Lambda}) d(\Delta T)\, d\vec{\Lambda} \qquad (42) \end{aligned}$$

To find the constant A we integrate with respect to $\vec{\Lambda}$ and use the fact that $f_{cl}(\vec{\Lambda})$ is a density, so that its integral over the whole $\vec{\Lambda}$-range must be equal to 1. We then get

$$A^{-1} = \int_{\vec{\Lambda}} \int_0^\infty (\Delta T) f(\Delta T,\vec{\Lambda}) d(\Delta T) d\vec{\Lambda} = \mu_{\Delta T}$$

where $\mu_{\Delta T}$ is the mean value of the random variable ΔT. Inserting the latter into (42) we obtain equation (39), which had to be proved.

Let us conclude this section with a note of warning. Since $f_{cl}(\vec{\Lambda})$ is the first-order pdf of the stationary stochastic process $\vec{\Lambda}(\tau)$, it should be estimated in practice by a suitable estimator such as [Bendat & Piersol (1971)]:

$$f_{cl}(\vec{\Lambda})d\vec{\Lambda} \cong \frac{\begin{array}{l}\tau\text{-time spent by } \vec{\Lambda}(\tau) \text{ in}\\ [\vec{\Lambda},\vec{\Lambda} + \Delta\vec{\Lambda}) \text{ during a}\\ \tau\text{-interval } [0,T_1]\end{array}}{T_1}$$

An empirical distribution $f_{sd}(\vec{\Lambda})$, obtained by methods similar to that used for constructing the standard scatter diagram $f_{sd}(H_S,T_0)$, should not, in principle, be considered as a satisfactory approximation of $f_{cl}(\vec{\Lambda})$. For the scatter diagrams are obtained by measuring every k hours, k usually ranging from 3 to 12, or even being random, so that $f_{sd}(\vec{\Lambda})$ will be biased towards the long-duration sea-states (length-biased sampling). However, replacing $f_{cl}(\vec{\Lambda})$ by $f_{sd}(\vec{\Lambda})$ is the best we can do at the present time.

8. Concluding remarks—main steps of the model

The purpose of this paper was twofold: First, to construct a rational yet flexible model for long-term stochastic analysis of sea (wind) waves, that is, a model based on clearly defined notions and explicitly stated assumptions which encompasses previous works as special cases. Second, to properly model the variability of the duration of sea states in long-term stochastic calculations. Due to the complicated structure of the derived model, and in order to make clear its full flexibility and generality, it seems worthwhile to conclude the paper by presenting schematically and in a non-formal way the main steps of its construction. This presentation will be done separately, first for the conceptual framework of the two-level long-term stochastic model, and then for the specific procedure used in calculating the probability distribution of an additive long-term random quantity. Special attention will be paid to making clear the underlying assumptions made in each step.

We start with a non-formal description of the conceptual framework of the proposed stochastic model.

1. Consider the time history $\eta(t)$, $0 \leq t \leq T$, of the free-surface elevation at a given site, where $[0,T]$ is a long-term period.

2. By considering successive t-intervals (of order of one hour), we obtain the corresponding sequence of spectral characteristics $\vec{\Lambda}_i$, $i = 1,2, \ldots, M$. Within each such t-interval, the surface elevation $\eta(t)$ is modeled as a stationary stochastic process. This is the usual short-term or first-level stochastic modeling. More specific assumptions also should be imposed concerning the probability laws of the short-term process. Usually we use normality. However, any other relevant probability law can be considered [Longuet-Higgins (1963), Huang & Long (1980), Tayfun (1980,1981), Huang (1983), Tayfun (1984)]. Such a choice affects only the steps (f) and (g) of the procedure used for calculating $M_u(T,\beta)$ (see below).

3. By interpolation we obtain the time history of the spectral parameters $\vec{\Lambda}(\tau)$, $0 \leq \tau \leq T$, where τ is again the time, but in a scale coarser than the t-scale (that means: $d\tau \leftrightarrow$ large t-interval). $\vec{\Lambda}(\tau)$ is now modeled as a new stochastic process. This is the long-term or second-level stochastic modeling. Again specific assumptions should be imposed in order to describe the stochastic nature of $\vec{\Lambda}(\tau)$. These assumptions should make a compromise between theoretical complexity and physical reality. In this connection $\vec{\Lambda}(\tau)$ can be modeled as a stationary stochastic process, a seasonally stationary stochastic process, a periodically correlated stochastic process, etc. As regards the probability laws characterizing $\vec{\Lambda}(\tau)$, various models can be used. The log-normal distribution seems a promising choice.

4. Using the continuous τ-time process $\vec{\Lambda}(\tau)$ and the definition of a sea state given in Section 3, we then obtain the stochastic sequence of the successive sea states $\vec{\Lambda}_i$ and their durations ΔT_i: $(\vec{\Lambda}_1,\Delta T_1)$, $(\vec{\Lambda}_2,\Delta T_2)$, ... The stochastic nature of this sequence can be either derived theoretically, on the basis of the stochastic properties of the process $\vec{\Lambda}(\tau)$, or estimated statistically through direct analysis of appropriate long-term data. In the second case, which is followed in this paper, it is necessary to impose an ad hoc specific stochastic structure on the sequence $(\vec{\Lambda}_i,\Delta T_i)$ $i = 1,2, \ldots$ Possible choices are the renewal process or the semi-Markov process.

On the basis of the above comments one can see that the assumptions concerning the long-term stochastic structure of the model can be imposed either on the continuous τ-time process $\vec{\Lambda}(\tau)$, or on the sequence $(\vec{\Lambda}_i,\Delta T_i)$, $i = 1,2, \ldots$.

Using the above-described model we are able to pose definite questions and give complete quantitative answers concerning various long-term quantities. (This is the fundamental novelty of the present work). The main steps for calculating the probability distributions of an additive long-term quantity are described below. For the sake of definiteness we use the specific quantity $M_u(T,\beta)$ = "Number of peaks of the surface elevation at a specific point, lying above the level u and occurring in a long-term interval $[0,T]$."

(a) Using the *additivity* of $M_u(T,\beta)$ we represent the long-term quantity $M_u(T,\beta)$ as a sum of corresponding short-term quantities, each of which is related to one sea state $\Xi_i(\beta) = ([T_{bi}(\beta),T_{ei}(\beta)], \vec{\Lambda}_i(\beta))$

$$M_u(T,\beta) = \sum_{i=1}^{N(T;\beta)} M_u(\Xi_i(\beta);\beta)$$

(b) Using the *hierarchy assumption* we estimate the number $M_u(\Xi_i(\beta);\beta)$ through the relation

$$M_u(\Xi_i(\beta);\beta) \cong \mathbf{E}^{\gamma_i}[M_u^{ST}(\Xi_i(\beta);\gamma_i)]$$

that is, as a short-term mean value of the corresponding short-term quantity.

(c) Using the *short-term stationarity assumption* we set

$$\mathbf{E}^{\gamma_i}[M_u^{ST}(\Xi_i(\beta);\gamma_i)] = \Delta T_i(\beta)W(\vec{\Lambda}_i(\beta)) = M_i(\beta)$$

where $W(\vec{\Lambda}_i(\beta))$ is the corresponding short-term quantity per unit time.

(d) On the basis of the *long-term seasonal stationarity* and the *independence assumption* the sequence $(\Delta T_i(\beta),M_i(\beta))$, $i = 1,2, \ldots$ is given the structure of a renewal-reward process. Alternative choices are also possible. For example, the sequence $(\Delta T_i(\beta),M_i(\beta))$, $i = 1,2, \ldots$ might be considered as a semi-Markov process.

(e) Using the asymptotic theory of renewal-reward processes we find that the probability law of $M_u(T)$ is Gaussian, and we calculate its parameters. See Section 5, relations (30), (31). (This is the most fundamental new result given in this work.)

(f) Using the assumption of *short-term normality* we can apply formula (46) obtaining the result (33).

(g) Finally, using the assumption of narrow-band sea states, we end in the expression (34) for the long-term mean value of $M_u(T)$, which is actually the same with Battjes's (1970) result. [See also Ochi (1982).]

From the above exposition it becomes clear that the proposed long-term stochastic model is rather a methodology than a theory, and it can be used in combination with various different specific stochastic models, whichever seems more suitable (and tractable) per step. In this connection we emphasize that only the short-term stationarity and the hierarchy assumption are indispensable, while all other ones can be changed in various ways.

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Appendix 1

Kind of events which can be treated by present theory

Consider a random quantity (point process)[24] $\chi(S;\beta)$ defined on the primary stochastic process $\{\eta(t,\beta), \beta \in B\}$. Here, as in Section 5, β is a choice

[24]The concept of a point process came back to Palm (1943), who also developed the basic theory. Palm's ideas were developed further and made rigorous by Khintchine in 1960 in his classical work on queuing theory. For a rigorous modern account of the general theory of point processes and their natural extensions, the random measures, see Kallenberg (1976), Matthes et al (1978), Cox & Isham (1980).

variable denoting the sample function used to evaluate $\chi(S;\beta)$, and S is a subset of the time axis during which the evaluation process takes place. In this Appendix we list the conditions which should satisfy $\chi(S;\beta)$ in order to be amenable to the treatment presented in Section 5 of this paper. These are the following:

(C1) *The extensibility condition*—$\chi(S;\beta)$ is well defined for any time interval $S = [\tau_1,\tau_2]$, either in the short- or in the long-term range.

(C2) *The additivity condition*—$\chi(S;\beta)$ is non-negative and additive with respect to its set variable S, that is, $\chi(S_1 \cup S_2;\beta) = \chi(S_1;\beta) + \chi(S_2;\beta)$, whenever S_1 does not overlap S_2.

(C3) *The hierarchy condition*—Let $S_i(\beta) = [T_{bi}(\beta),T_{ei}(\beta)]$ be the time interval corresponding to the ith individual sea state, and γ_i denote the sample functions of the stationary stochastic process corresponding to that sea state (recall the construction of Section 4). Then, the number $\chi(S_i(\beta);\beta)$ can be approximated by means of the short-term mean value $\mathbf{E}^{\gamma_i}[\chi(S_i(\beta);\gamma_i)]$, and thus the mean value $\mathbf{E}^{\beta}[\chi(S_i(\beta);\beta)]$ can be calculated with the aid of the following two-level hierarchy model

$$\mathbf{E}^{\beta}\left[\sum_i \mathbf{E}^{\gamma_i}[\chi(S_i(\beta);\gamma_i)]\right]$$

Examples of useful events which may be considered satisfying the above three conditions are:

1. The number of maxima above a given level,
2. The time spent above a given level,
3. The number of upcrossings of a given level,
4. The number of times in which n successive maxima occur above a given level,
5. The number of maxima above a given level for which the absolute value of the second derivative is greater than a given quantity.

Appendix 2

Some results from the theory of stationary stochastic processes

In this Appendix we report (in our notation) some standard results used in the main part of the paper concerning sample-function crossing problems for a real-valued, zero-mean, stationary and normal process $\eta(t)$, with twice differentiable sample paths. Most of results of this type have been first derived by Rice in 1944 and can be now found in many standard books and monographs as, for example, Ochi & Bolton (1973), Price & Bishop (1974), Ochi (1982), Middletton (1960), Cramer & Leadbetter (1967), Lin (1967).

The CDF of the zero-to-peak wave amplitude a is given by

$$G(a;\sigma,\varepsilon) = \Phi\left[\frac{a}{\varepsilon\sigma}\right] - \sqrt{2\pi}\,\delta\,\varphi(a/\sigma)\,\Phi\left[\frac{a\delta}{\varepsilon\sigma}\right] \tag{43}$$

where

$$\sigma = \sqrt{m_0},\ \varepsilon = \left(1 - \frac{m_2^2}{m_0 m_4}\right)^{1/2} \text{ and } \delta = \sqrt{1-\varepsilon^2} \tag{44}$$

where m_k is the kth spectral moment, and $\varphi(.)$, $\Phi(.)$ denote the pdf and CDF of the standardized normal distribution, respectively. For the special cases $\varepsilon \cong 0$ (narrow-band process) the above formula becomes

$$G(a;\sigma,\varepsilon = 0) = 1 - \exp\left[-\frac{a^2}{2\sigma^2}\right] \tag{45}$$

while for $\varepsilon > 0$ and $u = 0$ we obtain $G(0;\sigma,\varepsilon) = (1 - \delta)/2$.

The mean number of peaks (local maxima) of the process per unit time lying above a given level u is given by the formula

$$M(u;1,\vec{\Lambda}) \equiv W(u;\vec{\Lambda}) = \frac{1 - G(u;\sigma,\varepsilon)}{\delta\,T_0} \tag{46}$$

where $\vec{\Lambda} = (\sigma,T_0,\varepsilon)$, and $T_0 = 2\pi\sqrt{m_0/m_2}$ is the mean zero-upcrossing period of the process $\eta(t)$. For $u = 0$, equation (46) gives the mean number of positive peaks of the process per unit time $M^+(1,\vec{\Lambda}) = (1 + \delta)/2\delta T_0$.