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NATURAL DEDUCTION FOR FIRST-ORDER PURE IMPERATIVE LOGIC*

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Abstract

First-Order Pure Imperative Logic (FOPIL) deals with arguments from imperative premises to imperative conclusions (i.e., pure imperative arguments) that may contain quantifiers and identity. FOPIL can be used to symbolize, for example, the reasoning from "close the door of every office in the basement" to "if your office is in the basement, close its door". I present a natural deduction system for FOPIL that consists of replacement and inference rules that represent natural patterns of reasoning. I prove that two imperative formulas are logically equivalent exactly if one of them can be derived from the other by means of replacement rules, and that a pure imperative argument is valid exactly if its conclusion can be derived from its premises by means of replacement or inference rules.

Keywords: imperative logic, natural deduction, logical equivalence, argument validity

1. Introduction

Here is a logic test for you. Symbolize sentences (1) and (2) below by using the provided symbols, and then show by natural deduction that the corresponding argument is valid:

(1) Obey the laws, but if and only if they are just.
Thus: (2) If a law is not just, do not obey it.
(*Lx*: *x* is a law; *Ox*: you obey *x*; *Jx*: *x* is just.)

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You may complain that this test is unfair: you were never taught how to symbolize quantified imperative English sentences like (1) or how natural deduction applies to arguments with quantified imperative premises or conclusions. The complaint is reasonable: these topics are neglected in the literature.[1] In this paper, I take steps to remedy this neglect. I present a natural deduction system for *First-Order Pure Imperative Logic* (FOPIL), which deals with arguments from imperative premises to imperative conclusions (i.e., pure imperative arguments) that may contain quantifiers and identity. First, I introduce a formal language that can be used to symbolize quantified imperative English sentences (§2). Second, I introduce interpretations of the formal language, and I use them to define the logical equivalence of imperative formulas (§3). Third, I define the validity of a pure imperative argument (§4). Fourth, I provide a set of replacement rules such that (as I prove in the Appendix) two imperative formulas are logically equivalent exactly if one of them can be derived from the other by means of the replacement rules (§5). Fifth, I provide a set of inference rules such that (as I prove in the Appendix) a pure imperative argument is valid exactly if its conclusion can be derived from its premises by means of the replacement and inference rules (§6). This paper continues my previous work on imperative logic (Vranas 2008; 2010; 2011; 2013; 2016; 2019); I refer frequently to parts of that work, but I do not presuppose any familiarity with those parts.

2. Syntax

The (imperative formal) language of FOPIL has the following symbols: the connectives '~', '&', '$\vee$', '$\rightarrow$', and '$\leftrightarrow$', the punctuation symbols '(' and ')', the *imperative operator* '!' ("let it be the case that"), the quantifiers '$\forall$' and '$\exists$', the identity symbol '=', the variables '*x*', '*y*', '*z*', 'x'', 'y'', 'z'', 'x''', …, the constants '*a*', '*b*', '*c*', '*d*', 'a'', …, and, for any $n \geq 0$, the *n*-place predicates 'A_n', 'B_n', …, 'Z_n', 'A'_n', …. The *terms* (of the language of FOPIL) are the variables plus the constants. The *atomic formulas* are the strings of symbols $\ulcorner(f_1 = f_2)\urcorner$ and $\ulcorner\Pi f_1 \ldots f_n\urcorner$ for any terms $f_1, \ldots, f_n$ and any *n*-place predicate Π. The *declarative formulas* are all and only those finite strings of symbols that either are atomic formulas or can be built up from atomic formulas by applying at least once the following formation rule: if *p* and *q* are declarative formulas and *u* is a variable, then $\ulcorner{\sim}p\urcorner$, $\ulcorner(p \,\&\, q)\urcorner$, $\ulcorner(p \vee q)\urcorner$, $\ulcorner(p \rightarrow q)\urcorner$, $\ulcorner(p \leftrightarrow q)\urcorner$, $\ulcorner\forall up\urcorner$, and $\ulcorner\exists up\urcorner$ are declarative formulas. The *imperative formulas* are all and only those finite strings of symbols

[1] Only Hansen 2014 and Vranas 2019 provide sound and complete natural deduction systems for imperative logic, but their formal languages do not include quantifiers or identity. See also Fine 2018, 625–626.

that can be built up from declarative formulas by applying the following formation rules (R1 must be applied at least once):

(R1) If p is a declarative formula, then $\ulcorner !p \urcorner$ is an imperative formula.
(R2) If i and j are imperative formulas, then $\ulcorner {\sim} i \urcorner$, $\ulcorner (i \,\&\, j) \urcorner$, and $\ulcorner (i \vee j) \urcorner$ are also imperative formulas.
(R3) If p is a declarative formula and i is an imperative formula, then $\ulcorner (p \rightarrow i) \urcorner$, $\ulcorner (i \rightarrow p) \urcorner$, $\ulcorner (p \leftrightarrow i) \urcorner$, and $\ulcorner (i \leftrightarrow p) \urcorner$ are imperative formulas.
(R4) If u is a variable and i is an imperative formula, then $\ulcorner \forall u i \urcorner$ and $\ulcorner \exists u i \urcorner$ are imperative formulas.

A *formula* is either a declarative formula or an imperative formula. It follows from these definitions that a formula is imperative exactly if it contains at least one occurrence of '!' and is declarative exactly if it contains no occurrence of '!' (so no formula is both declarative and imperative). A *subformula* of a given formula is any string of consecutive symbols of the given formula that is itself a formula. An occurrence of '!' or of a variable u in a formula φ is *bound* in φ exactly if it is also an occurrence of '!' or u in a subformula of φ that begins with $\ulcorner \forall u \urcorner$ or with $\ulcorner \exists u \urcorner$, and is *free* in φ otherwise. A (*declarative*, *imperative*, or *atomic*) formula is *closed* (i.e., is a *sentence*) exactly if no occurrence of any variable in the formula is free. For simplicity, I usually omit subscripts and outermost parentheses.

Here are some examples of how imperative English sentences can be symbolized in FOPIL ('Vx' stands for "you vaccinate x", 'Nx' for "x is a neonate", 'a' stands for Alice, and so on):

(1) Vaccinate Alice: $!Va$
(2) Don't vaccinate Alice: ${\sim}!Va$
(3) Vaccinate Alice and Brenda: $!Va \,\&\, !Vb$
(4) Vaccinate every neonate: $\forall x(Nx \rightarrow !Vx)$
(5) Vaccinate only neonates: $\forall x(!Vx \rightarrow Nx)$
(6) Vaccinate some neonate: $\exists x(Nx \rightarrow !Vx)$
(7) Obey the laws, but if and only if they are just: $\forall x(Lx \rightarrow (!Ox \leftrightarrow Jx))$
(8) If a law is not just, do not obey it: $\forall x(Lx \rightarrow ({\sim}Jx \rightarrow {\sim}!Ox))$

I realize that (6) may look like a mistake, but I discuss it (together with (4)) in §5. Concerning (3), one might wonder why "vaccinate Alice and Brenda" was symbolized as '$!Va \,\&\, !Vb$' instead of '$!(Va \,\&\, Vb)$'. I reply that either symbolization will do: (a) '$!Va \,\&\, !Vb$' symbolizes "let it be the case that you vaccinate Alice, and let it be the case that you vaccinate Brenda", (b) '$!(Va \,\&\, Vb)$' symbolizes "let it be the case that you vaccinate both Alice and Brenda", (c) both English sentences are adequate paraphrases of "vaccinate Alice and Brenda",[2] and (d) it turns out (see §3) that '$!Va \,\&\, !Vb$' and

[2] Changing slightly the example, one might argue that (1) "let it be the case that Pat vaccinates Alice" is not an adequate paraphrase of (2) "Pat, vaccinate Alice" because (2) is

'!(*Va* & *Vb*)' are logically equivalent. Similarly, concerning (2), it turns out that '~!*Va*' ("let it not be the case that you vaccinate Alice") and '!~*Va*' ("let it be the case that you don't vaccinate Alice") are logically equivalent.[3] Note that '~!*Va*' is a *negation*, namely a formula of the form ⌜~φ⌝ (where φ is a declarative or imperative formula), but '!~*Va*' is what I call an *unconditionally prescriptive formula*, namely a formula of the form ⌜!*p*⌝ (where *p* is a declarative formula).

One might wonder why I use a single set of connectives both for declarative and for imperative logical operations; why not use instead, for example, '&' for declarative conjunction (as in '!(*Va* & *Vb*)') and '$\&_{\mathrm{i}}$' for imperative conjunction (as in '!*Va* $\&_{\mathrm{i}}$!*Vb*')? I reply that the proliferation of connectives would make the notation cumbersome. Note that English likewise uses a single set of coordinating conjunctions both for declarative and for imperative (syndetic) coordination, as the two occurrences of "and" in "if you marry and ovulate, then copulate and procreate" illustrate. One might argue that this ambiguity is an undesirable feature of English and should be eliminated in a formal language. I reply that my use of a single set of connectives does not result in any confusion: it is always clear whether (for example) the ampersand is connecting declarative or imperative formulas, and semantically (see §3) the ampersand is treated differently in the two cases.

According to the first formation rule for imperative formulas (R1), if *p* is *any* declarative formula, then ⌜!*p*⌝ is an imperative formula. This is as it should be, because prefixing *any* declarative English sentence with "let it be the case that" yields an imperative English sentence. For example, "let it be the case that last week I died" is an imperative English sentence, even if one that would hardly ever be used (cf. Vranas 2008, 555 n. 17).[4]

addressed to Pat but (1) is not. In reply, consider the following parallel argument concerning *declarative* sentences: (3) "I predict that Pat will vaccinate Alice" is not an adequate paraphrase of (4) "Pat, I predict that you will vaccinate Alice" because (4) is addressed to Pat but (3) is not. This argument fails: (3) *is* an adequate paraphrase of (4), in the sense that (3) and (4) normally express the same *proposition*. Similarly, (1) *is* an adequate paraphrase of (2), in the sense that —in my preferred terminology—(1) and (2) normally express the same *prescription* (as I argue in Vranas 2008, 554 n. 14, n. 15).

[3] Cf. Parsons 2013, 84–85; contrast Clark-Younger & Girard 2013. Imperative English sentences have *imperative negations* (which are also imperative English sentences; e.g., an imperative negation of "pay" is "don't pay"), but one might argue that they also have *permissive negations* (which are permissive English sentences; e.g., one might argue that a permissive negation of "pay" is "you may fail to pay"). I do not deal with permissive sentences in this paper.

[4] In case one thinks that R1 is too permissive (because English sentences like "let it be the case that last week I died" and "let it be the case that the Earth revolves around the Sun" should be excluded from consideration; see, e.g., Rescher 1966, 34), one can (1) designate some predicates as *agential and future-directed* predicates, (2) designate those declarative formulas that contain only such predicates as *agential and future-directed* declarative formulas, and (3) replace R1 with: (R1′) if *p* is an agential and future-directed declarative

By contrast, certain strings of symbols of FOPIL are *not* formulas. (1) If i is an imperative formula, then $\ulcorner !i \urcorner$ is *not* a formula. For example, '$!!Va$' is not a formula. This is as it should be, because "let it be the case that let it be the case that you vaccinate Alice" is not an English sentence; more generally, prefixing an *imperative* English sentence with "let it be the case that" does not yield an English sentence.[5] (2) If i and j are imperative formulas, then $\ulcorner(i \rightarrow j)\urcorner$ and $\ulcorner(i \leftrightarrow j)\urcorner$ are *not* formulas (contrast Castañeda 1975, 113–115). For example, '$(!Va \rightarrow !Vb)$' and '$(!Va \leftrightarrow !Vb)$' are not formulas. This is as it should be, because "if vaccinate Alice, vaccinate Brenda" (or "vaccinate Alice only if vaccinate Brenda") and "if and only if vaccinate Alice, vaccinate Brenda" (or "vaccinate Alice if and only if vaccinate Brenda") are not English sentences. (3) If p is a declarative formula and i is an imperative formula, then $\ulcorner(p \;\&\; i)\urcorner$, $\ulcorner(i \;\&\; p)\urcorner$, $\ulcorner(p \vee i)\urcorner$, and $\ulcorner(i \vee p)\urcorner$ are *not* formulas (cf. Vranas 2008, 560 n. 41; contrast Fox 2012, 885–886). For example, '$({\sim}Va \;\&\; !Vb)$' is not a formula. This may seem undesirable, because "although you are not going to vaccinate Alice, at least vaccinate Brenda" *is* an English sentence. I reply that nothing important is lost by symbolizing the two parts of the English sentence separately, as '${\sim}Va$' and '$!Vb$'.[6] Counting '$({\sim}Va \;\&\; !Vb)$' as a formula would complicate the syntax without yielding any commensurate benefit. I discuss this issue further in §5.[7]

formula, then $\ulcorner !p \urcorner$ is an imperative formula. (An alternative approach would be to define *action formulas*, corresponding to actions, and to prefix only action formulas, not declarative formulas, with '!': see Segerberg 1990.)

[5] One might ask: why should English be the adjudicator of a formal language? I answer that I am interested in a formal language that corresponds to the way we use imperative (and declarative) sentences in English (and other natural languages). One might claim that (1) "let it be the case that you (will) let it be the case that you vaccinate Alice" *is* an English sentence, and one might take this as a reason to count '$!!Va$' as a formula logically equivalent to '$!Va$' (cf. Chellas 1971, 124–125). I agree that (1) is an English sentence, but this is not a reason to count '$!!Va$' as a formula, because "let it be the case that", understood *impersonally*, does *not* occur twice in (1): only the first occurrence of "let" in (1) is impersonal. (Note that "you (will) let it be the case that you vaccinate Alice"—in contrast to "let it be the case that you vaccinate Alice"—is a *declarative* English sentence.)

[6] One might object that my reply does not work for disjunctions: it will not do to symbolize separately the two parts of the English sentence (1) "punish Brenda, or I will". I respond that this English sentence can be paraphrased as (2) "if you don't punish Brenda, then I will" (or alternatively as (3) "punish Brenda; if you don't, then I will"), so to symbolize the English sentence no disjunction of an imperative with a declarative formula is needed. (Following Starr (2020, 8), one might argue that, if (1) is felicitously followed by "I don't care which", then (1) cannot be paraphrased as (3). I reply that in such a case (1) can still be paraphrased as (2).)

[7] To my knowledge, the imperative operator was introduced into formal languages by Mally (1926; see also Hofstadter & McKinsey 1939). Two alternatives to R1 have been proposed in the literature. (1) Clarke and Behling (1998, 282–284; cf. Clarke 1973, 192–193) propose *postfixing* (instead of prefixing) sentence letters (i.e., 0-place predicates) with '!' to form imperative sentences. To allow for example '$(A \rightarrow B)!$'—in addition to '$A \rightarrow B!$'—to count as a sentence, generalize the proposal as follows: postfixing *any* declarative sentence

3. Semantics

An *interpretation* of the language of FOPIL is an ordered triple whose three components are:

- First, a non-empty set (called the *domain* of the interpretation).
- Second, a function (called the *denotation function* of the interpretation) that (1) assigns to every constant a member of the domain (the *referent* of the constant on the interpretation), (2) assigns to every sentence letter (i.e., 0-place predicate) either ‘T’ or ‘F’ (the *truth value* of the sentence letter on the interpretation), and (3) assigns to every *n*-place predicate (for $n \geq 1$) a set of ordered *n*-tuples of members of the domain (the *extension* of the predicate on the interpretation).
- Third, a three-place relation on declarative sentences (called the *favoring relation* of the interpretation) that satisfies the following two conditions. (1) The *intensionality condition*: for any declarative sentences *p*, *q*, and *r*, and any declarative sentences p', q', and r' interderivable in CFOL (i.e., classical first-order logic) with *p*, *q*, and *r* respectively, the ordered triple $\langle p, q, r\rangle$ is in (i.e., is a member of) the relation exactly if $\langle p', q', r'\rangle$ is. (2) The *asymmetry condition*: for any declarative sentences *p*, *q*, and *r*, $\langle p, q, r\rangle$ and $\langle p, r, q\rangle$ are not both in the relation.

Say that *p favors q over r* on an interpretation exactly if $\langle p, q, r\rangle$ is in the favoring relation of the interpretation. Then, to say that the favoring relation of any interpretation satisfies the asymmetry condition is to say that, for any declarative sentences *p*, *q*, and *r*, *p* does not favor both *q* over *r* and *r* over *q* on any interpretation. Informally, a favoring relation corresponds to *comparative reasons* (e.g., reasons for you to marry Hugh *rather than* Hugo), so the asymmetry condition corresponds to the claim that nothing can be a reason both for *q* rather than *r* and for *r* rather than *q*. In this paper, I do

(not just sentence letters) with ‘!’ yields an imperative sentence. But then is ‘~*A*!’ a negation or an unconditionally prescriptive sentence? Resolving the ambiguity by saying that, if φ is a sentence, then ⌜(~φ)⌝ (instead of ⌜~φ⌝) is a sentence (so that ‘~*A*!’—i.e., ‘(~*A*!)’ with the outermost parentheses omitted—is a negation but ‘(~*A*)!’ is an unconditionally prescriptive sentence) results in a proliferation of parentheses, as for example in ‘(~*A*)! & (~*B*)!’—which corresponds in my notation to ‘!~*A* & !~*B*’. Moreover, the proposal introduces an unwelcome asymmetry between modal (and other sentential) operators, which are prefixed, and the imperative operator, which is postfixed. (2) Gensler (1990, 190; 1996, 182; 2002, 184) proposes *underlining* sentence letters (instead of prefixing them with ‘!’) to form imperative sentences. To allow for example ‘$\underline{A \rightarrow B}$’—in addition to ‘$A \rightarrow \underline{B}$’—to count as a sentence, generalize the proposal as follows: underlining *any* declarative sentence (not just sentence letters) yields an imperative sentence. This proposal results in a proliferation of underlining, as for example in ‘$\underline{(A \,\&\, (B \vee C))} \rightarrow D$’—which corresponds in my notation to ‘!$(A \,\&\, (B \vee C)) \rightarrow D$’. Moreover, the proposal results in a proliferation of symbols: if sentences are ordered *n*-tuples of symbols and ‘$\underline{A \rightarrow B}$’ is a sentence, then ‘$\underline{A}$’, ‘$\underline{\rightarrow}$’, and ‘$\underline{B}$’ are symbols.

not need to engage with metaphysical debates concerning the existence and the nature of reasons for action (see, e.g., Skorupski 2010): the above formal definition of a favoring relation suffices for my purposes. The favoring relation is used in §4 to define the validity of pure imperative arguments.

Declarative sentences are *true* or *false* on interpretations, and imperative sentences are *satisfied*, *violated*, or *avoided* on interpretations. Specifically, for any sentence letter *e*, any constants $h_1, \ldots, h_n$, any *n*-place predicate Π (for $n \geq 1$), any declarative sentences *p* and *q*, any imperative sentences *i* and *j*, and any interpretation *m*:

Truth and falsity of a declarative sentence on an interpretation

(D1) *e* is true on *m* iff (i.e., exactly if) *m* assigns 'T' to *e*.

(D2) $\ulcorner h_1 = h_2 \urcorner$ is true on *m* iff the referent of h_1 on *m* is the same as the referent of h_2 on *m*.

(D3) $\ulcorner \Pi h_1 \ldots h_n \urcorner$ is true on *m* iff the ordered *n*-tuple whose components are the referents of $h_1, \ldots, h_n$ on *m* (in that order) is a member of the extension of Π on *m*.

(D4) $\ulcorner \sim p \urcorner$ is true on *m* iff *p* is not true on *m*.

(D5) $\ulcorner p \,\&\, q \urcorner$ is true on *m* iff both *p* and *q* are true on *m*.

(D6) $\ulcorner p \vee q \urcorner$ is true on *m* iff $\ulcorner \sim(\sim p \,\&\, \sim q) \urcorner$ is true on *m*.

(D7) $\ulcorner p \rightarrow q \urcorner$ is true on *m* iff $\ulcorner \sim p \vee q \urcorner$ is true on *m*.

(D8) $\ulcorner p \leftrightarrow q \urcorner$ is true on *m* iff $\ulcorner (p \rightarrow q) \,\&\, (q \rightarrow p) \urcorner$ is true on *m*.

(D9) *p* is false on *m* iff *p* is not true on *m*.

Satisfaction, violation, and avoidance of an imperative sentence on an interpretation

(I1) $\ulcorner !p \urcorner$ is (a) satisfied on *m* iff *p* is true on *m*, and is (b) violated on *m* iff *p* is false on *m*. (Informally: "vaccinate Alice" is satisfied iff you vaccinate Alice, and is violated iff you do not vaccinate Alice.)

(I2) $\ulcorner \sim i \urcorner$ is (a) satisfied on *m* iff *i* is violated on *m*, and is (b) violated on *m* iff *i* is satisfied on *m*. (Informally: "don't vaccinate Alice" is satisfied iff you do not vaccinate Alice, namely iff "vaccinate Alice" is violated, and is violated iff you vaccinate Alice, namely iff "vaccinate Alice" is satisfied.)

(I3) $\ulcorner i \,\&\, j \urcorner$ is (a) satisfied on *m* iff (i) *i* is satisfied and *j* is not violated on *m* or (ii) *j* is satisfied and *i* is not violated on *m*, and is (b) violated on *m* iff at least one of *i* and *j* is violated on *m*. (Note that $\ulcorner i \,\&\, j \urcorner$ can be satisfied on *m* even if not both *i* and *j* are satisfied on *m*. Informally: "if you vaccinate Alice, vaccinate Brenda, and if you don't vaccinate Alice, vaccinate Brenda" is equivalent to "vaccinate Brenda (regardless of whether you vaccinate Alice)", so it is satisfied if you vaccinate Brenda but not Alice and thus if "if you vaccinate Alice, vaccinate Brenda" is not satisfied (see I5 below).)

(I4) $\ulcorner i \vee j \urcorner$ is satisfied (or violated) on *m* iff $\ulcorner {\sim}({\sim}i \,\&\, {\sim}j) \urcorner$ is satisfied (or violated) on *m*. (Informally: "vaccinate Alice or Brenda" is equivalent to the negation of "vaccinate neither Alice nor Brenda".)

(I5) $\ulcorner p \rightarrow i \urcorner$ is (a) satisfied on *m* iff both *p* is true on *m* and *i* is satisfied on *m*, and is (b) violated on *m* iff both *p* is true on *m* and *i* is violated on *m*. (Informally: "if you vaccinate Alice, vaccinate Brenda" is satisfied iff you vaccinate both Alice and Brenda, and is violated iff you vaccinate Alice but not Brenda.)

(I6) $\ulcorner i \rightarrow p \urcorner$ is satisfied (or violated) on *m* iff $\ulcorner {\sim}p \rightarrow {\sim}i \urcorner$ is satisfied (or violated) on *m*. (Informally: "vaccinate Alice only if you vaccinate Brenda" is equivalent to "if you don't vaccinate Brenda, don't vaccinate Alice".)

(I7) $\ulcorner p \leftrightarrow i \urcorner$ is satisfied (or violated) on *m*—and the same holds for $\ulcorner i \leftrightarrow p \urcorner$—iff $\ulcorner (p \rightarrow i) \,\&\, (i \rightarrow p) \urcorner$ is satisfied (or violated) on *m*. (Informally: "if and only if you vaccinate Alice, vaccinate Brenda"—i.e., "vaccinate Brenda if and only if you vaccinate Alice"—is equivalent to "vaccinate Brenda if you vaccinate Alice, and vaccinate Brenda only if you vaccinate Alice".)

(I8) *i* is avoided on *m* iff *i* is neither satisfied nor violated on *m*. (Informally: "vaccinate Alice" is avoided iff you neither vaccinate Alice nor do not vaccinate Alice, namely never, and "if you vaccinate Alice, vaccinate Brenda" is avoided iff you neither both vaccinate Alice and Brenda nor vaccinate Alice but not Brenda, namely iff you do not vaccinate Alice.)

See Vranas 2008, 532–545 for a detailed defense of I1–I8. There are also four rules that deal with quantifiers. To formulate them, I introduce first some notation and terminology. Take any variable *u*, any constant *h*, any member *o* of the domain of *m*, and any formula φ in which no occurrence of any variable different from *u* is free. Let φ[*u*/*h*] be the sentence that results from replacing in φ every occurrence of *u* that is free in φ with *h*. (If φ is a sentence, then φ[*u*/*h*] is just φ.) Let *m*[*h*/*o*] be the interpretation that results from replacing in *m* the referent of *h* with *o*. (So *m* and *m*[*h*/*o*] have the same domain and favoring relation, and their denotation functions differ only in the referent they assign to *h*. If the referent of *h* on *m* is *o*, then *m*[*h*/*o*] is just *m*.) If φ is a *declarative* formula, say that *o verifies* φ on *m* exactly if, for any (equivalently: for some) constant *h* that does not occur in φ, φ[*u*/*h*] is true on *m*[*h*/*o*]. If φ is an *imperative* formula, say that *o satisfies* φ on *m* exactly if, for any (equivalently: for some) constant *h* that does not occur in φ, φ[*u*/*h*] is satisfied on *m*[*h*/*o*], and define similarly what it is for *o* to *violate* or to *avoid* φ on *m*. Letting Δ be the domain of *m*, here are the four rules that deal with quantifiers:

(D10) $\ulcorner \forall up \urcorner$ is true on *m* iff every member of Δ verifies *p* on *m*.
(D11) $\ulcorner \exists up \urcorner$ is true on *m* iff some member of Δ verifies *p* on *m*.

(I9) ⌜$\forall ui$⌝ is (a) satisfied on *m* iff both some member of Δ satisfies *i* on *m* and no member of Δ violates *i* on *m*, and is (b) violated on *m* iff some member of Δ violates *i* on *m*.

(I10) ⌜$\exists ui$⌝ is (a) satisfied on *m* iff some member of Δ satisfies *i* on *m*, and is (b) violated on *m* iff both some member of Δ violates *i* on *m* and no member of Δ satisfies *i* on *m*.

In these four rules (in contrast to the previous rules), *p* and *i* need not be sentences: they must be formulas such that ⌜$\forall up$⌝, ⌜$\exists up$⌝, ⌜$\forall ui$⌝, and ⌜$\exists ui$⌝ are sentences (i.e., in these four rules, no occurrence of any variable different from *u* is free in *p* or in *i*). See Vranas 2008, 549–550 for a defense of I9 and I10 based on understanding universal and existential quantification as generalizations of conjunction and disjunction respectively. By I8–I10, ⌜$\forall ui$⌝ is avoided on *m* exactly if ⌜$\exists ui$⌝ is avoided on *m*, and also exactly if every member of Δ avoids *i* on *m*.

A *contradiction* is either a declarative sentence that is false on every interpretation (a *declarative contradiction*) or an imperative sentence that is violated on every interpretation (an *imperative contradiction*). Sentences φ and ψ are *logically equivalent* exactly if either (1) they are both declarative and they have the same truth value on every interpretation (i.e., for any interpretation *m*, φ and ψ are either both true on *m* or both false on *m*; equivalently, φ is true on *m* exactly if ψ is true on *m*) or (2) they are both imperative and they have the same "satisfaction value" on every interpretation (i.e., for any interpretation *m*, φ and ψ are either both satisfied on *m* or both violated on *m* or both avoided on *m*; equivalently, φ is satisfied on *m* exactly if ψ is satisfied on *m*, and φ is violated on *m* exactly if ψ is violated on *m*).[8] Moreover, one can define logical equivalence between formulas that need not be sentences: if φ and ψ are (either both declarative or both imperative) formulas and $u_1, \ldots, u_n$ are the only variables that have at least one free occurrence in φ or in ψ, then φ and ψ are *logically equivalent* exactly if, for any (equivalently: for some) distinct constants $h_1, \ldots, h_n$ that occur neither in φ nor in ψ, the sentences $\varphi[u_1/h_1, \ldots, u_n/h_n]$ and $\psi[u_1/h_1, \ldots, u_n/h_n]$ are logically equivalent (where $\varphi[u_1/h_1, \ldots, u_n/h_n]$ is the sentence that results from replacing in φ every occurrence of u_1 that is free in φ with h_1, and so on—and similarly for ψ). For example, '*Va* ∨ ~*Va*' is logically equivalent to '$x = x$' (although the former formula is a sentence but the latter one is not).

[8] Now it can be seen that, as stated in §2, '!*Va* & !*Vb*' and '!(*Va* & *Vb*)' are logically equivalent: for any interpretation *m*, '!*Va* & !*Vb*' is satisfied on *m* exactly if both '!*Va*' and '!*Vb*' are satisfied on *m* (by I3), and thus exactly if both '*Va*' and '*Vb*' are true on *m* (by I1), and thus exactly if '*Va* & *Vb*' is true on *m* (by D5), and thus exactly if '!(*Va* & *Vb*)' is satisfied on *m* (by I1)—and similarly for violation on *m*.

4. Strong and weak validity

A *pure imperative argument* is an ordered pair whose first component is a non-empty finite set of imperative sentences (the *premises* of the argument) and whose second component is an imperative sentence (the *conclusion* of the argument). Building on previous work, I say that (roughly) a pure imperative argument is valid when, on every interpretation, its conclusion is "supported" by everything that supports its premises. Also building on previous work (Vranas 2011; 2016; 2019), I distinguish *strong* from *weak* support—and, correspondingly, strong from weak validity—as follows:

> Definition 1. For any declarative sentence *p*, any imperative sentence *i*, and any interpretation *m*:
>
> (1) *p strongly supports i on m* exactly if (a) *p* is true on *m*, (b) *i* is not a contradiction, and (c) for any declarative sentences *q* and *r* that are not both contradictions, if (i) *i* is satisfied on every interpretation on which *q* is true and (ii) *i* is violated on every interpretation on which *r* is true, then *p* favors *q* over *r* on *m*;
> (2) *p weakly supports i on m* exactly if *p* strongly supports on *m* some imperative sentence *j* such that (a) *i* is satisfied on every interpretation on which *j* is satisfied and (b) *i* is avoided on all and only those interpretations on which *j* is avoided.
>
> Definition 2. A pure imperative argument is (1) *strongly valid* (i.e., its premises *strongly entail* its conclusion) exactly if, for any interpretation *m*, every declarative sentence that *strongly* supports on *m* every (equivalently: some) conjunction of all premises of the argument also *strongly* supports on *m* the conclusion of the argument, and is (2) *weakly valid* (i.e., its premises *weakly entail* its conclusion) exactly if, for any interpretation *m*, every declarative sentence that *weakly* supports on *m* every (equivalently: some) conjunction of all premises of the argument also *weakly* supports on *m* the conclusion of the argument.

Defending these definitions lies beyond the scope of this paper: I have extensively defended in previous work (Vranas 2011; 2016) an account of validity on which the definitions are based, and I have compared that account to alternative accounts in the literature (e.g., Charlow 2014; Kaufmann 2012; Parsons 2013; Segerberg 1990). Informally, the distinction between strong and weak validity captures a conflict of intuitions about whether, for example, "vaccinate Alice" entails "vaccinate or kill Alice": one can show that the pure imperative argument $\langle\{!Va\}, !(Va \vee Ka)\rangle$ is weakly but not strongly valid. (This argument corresponds to "Ross's paradox": see Ross 1941.)

5. Replacement derivations

In this section, I introduce a set of replacement rules such that two imperative formulas are logically equivalent exactly if one of them can be derived from the other by means of these replacement rules.

<u>Definition 3</u>. For any imperative formulas i and j, a *replacement derivation* of j from i is a finite sequence of imperative formulas (called the *lines* of the derivation) such that (1) the last line is j, (2) the first line is i, and (3) each line except the first can be obtained from the previous line by applying once a replacement rule from Table 1.

Name of rule and abbreviation		**Rule**		
Declarative Replacement	DR	If $p \dashv\vdash_{CFOL} q$,	then	$p \bowtie q$
Transposition[9]	TR	$i \rightarrow p$	$\bowtie$	$\sim p \rightarrow \sim i$
Negated Conditional	NC	$\sim(p \rightarrow i)$	$\bowtie$	$p \rightarrow \sim i$
Exportation	EX	$p \rightarrow (q \rightarrow i)$	$\bowtie$	$(p \mathbin{\&} q) \rightarrow i$
Commutativity	CO	$p \leftrightarrow i$	$\bowtie$	$i \leftrightarrow p$
Material Equivalence	ME	$p \leftrightarrow i$	$\bowtie$	$(p \rightarrow i) \mathbin{\&} (i \rightarrow p)$
Absorption	AB	$p \rightarrow\, !q$	$\bowtie$	$p \rightarrow\, !(p \mathbin{\&} q)$
Tautologous Antecedent	TA	$(p \vee \sim p) \rightarrow i$	$\bowtie$	i
Unconditional Negation	UN	$\sim !p$	$\bowtie$	$!\sim p$
Imperative Conjunction	IC	$(p \rightarrow\, !q) \mathbin{\&} (p' \rightarrow\, !q')$	$\bowtie$	$(p \vee p') \rightarrow\, !((p \rightarrow q) \mathbin{\&} (p' \rightarrow q'))$
Imperative Disjunction	ID	$(p \rightarrow\, !q) \vee (p' \rightarrow\, !q')$	$\bowtie$	$(p \vee p') \rightarrow\, !((p \mathbin{\&} q) \vee (p' \mathbin{\&} q'))$
Imperative Quantification	IQ	$\forall u(p \rightarrow\, !q)$ $\exists u(p \rightarrow\, !q)$	$\bowtie$ $\bowtie$	$\exists up \rightarrow\, !\forall u(p \rightarrow q)$ $\exists up \rightarrow\, !\exists u(p \mathbin{\&} q)$

Table 1. Replacement rules.

[9] One could have the following as a second part of Transposition: $\ulcorner p \rightarrow i \urcorner \bowtie \ulcorner \sim i \rightarrow \sim p \urcorner$. It is sometimes claimed in the literature, however, that imperative contraposition (or transposition) fails because, for example, "if you killed, confess" is not equivalent to "if you don't confess, let it not be the case that you killed". I reply that this is not really an instance of contraposition: the contrapositive of $\ulcorner \varphi \rightarrow \psi \urcorner$ is $\ulcorner \sim\psi \rightarrow \sim\varphi \urcorner$, so the contrapositive of '$K \rightarrow\, !C$' ("if you killed, confess") is '$\sim !C \rightarrow \sim K$' ("don't confess only if you didn't kill"; cf. Castañeda 1977, 780; Fox 2012, 892), not '$\sim C \rightarrow \sim !K$' ("if you don't confess, let it not be the case that you killed"). This is an example of how symbolization in an imperative formal language clears up a not uncommon confusion in the literature. I have been myself guilty of that confusion: in light of the above considerations, I renounce my definition of a contrapositive in Vranas 2011, 404 n. 45. See Vranas 2011, 404–405 n. 46 for references to imperative contraposition in the literature.

In Table 1, and in what follows, '$p \dashv\vdash_{CFOL} q$' abbreviates "p and q are interderivable in CFOL", and for any formulas φ and ψ, '$\varphi \bowtie \psi$' abbreviates "from any imperative formula k, one can obtain $k(\varphi/\psi)$ if φ is a subformula of k, and one can obtain $k(\psi/\varphi)$ if ψ is a subformula of k"—where $k(\varphi/\psi)$ is any formula that results from replacing in k at least one occurrence of φ with ψ. For simplicity, I omit corner quotes in tables.

Because the replacement rules are symmetric, there is a replacement derivation of j from i exactly if there is a replacement derivation of i from j. In the Appendix, I prove the following results: for any imperative formulas i and j, (a) if $i \bowtie j$ according to a replacement rule in Table 1, then i is logically equivalent to j, and (b) i is logically equivalent to j exactly if there is a replacement derivation of j from i.

All rules in Table 1 except Imperative Quantification (IQ) were also given in Vranas 2019.[10] Concerning IQ, note that, in the imperative formula '$\forall x(Nx \rightarrow !Vx)$', the only occurrence of '!' is bound (see §2). Since IQ (like every other rule in Table 1) corresponds to a logical equivalence, '$\forall x(Nx \rightarrow !Vx)$' is logically equivalent to '$\exists xNx \rightarrow !\forall x(Nx \rightarrow Vx)$'. But in the latter imperative formula, the only occurrence of '!' is free. It turns out that this holds in general: as I prove in the Appendix, for any imperative formula i, there are declarative formulas p and q such that i is logically equivalent to $\ulcorner p \rightarrow !q \urcorner$, so *every imperative formula is logically equivalent to some imperative formula (or other) in which no occurrence of '!' is bound* (since the only occurrence of '!' in $\ulcorner p \rightarrow !q \urcorner$ is free). It follows that, if one modified my definition of a formula in §2 by dropping the formation rule R4, there would still be enough formulas to symbolize every English sentence that can be symbolized in FOPIL. A grade of "imperative involvement" analogous to the "third grade of modal involvement" (Quine 1953) is redundant in FOPIL.

Going back now to some symbolizations I gave in §2, one might wonder why "vaccinate every neonate" was symbolized as '$\forall x(Nx \rightarrow !Vx)$'—equivalently, as '$\exists xNx \rightarrow !\forall x(Nx \rightarrow Vx)$'—instead of '$!\forall x(Nx \rightarrow Vx)$'. It turns out that '$!\forall x(Nx \rightarrow Vx)$' is *not* logically equivalent to '$\exists xNx \rightarrow$

[10] Note a disanalogy between declarative and imperative logic concerning Negated Conditional (NC): there is no general logical equivalence between negations of imperative conditionals and imperative conjunctions analogous to the general declarative logical equivalence between $\ulcorner \sim(p \rightarrow q) \urcorner$ and $\ulcorner p \;\&\; \sim q \urcorner$. On the contrary, negating a conditional with an imperative consequent amounts to negating the consequent: informally, "if you vaccinate Alice, don't vaccinate Brenda" negates "if you vaccinate Alice, vaccinate Brenda". Note that '$(Va \rightarrow !Vb) \;\&\; \sim(Va \rightarrow !Vb)$', which (as one can show) is logically equivalent to '$Va \rightarrow !(Vb \;\&\; \sim Vb)$', is *not* an imperative contradiction, namely an imperative sentence that is *violated* on every interpretation (cf. Charlow 2014, 628), but is instead what may be called an *imperative conditional contradiction*, namely an imperative sentence that is *non-satisfied* (i.e., violated or avoided) on every interpretation.

$!\forall x(Nx \rightarrow Vx)$': if there are no neonates, then '$!\forall x(Nx \rightarrow Vx)$'—which symbolizes (1) "let it be the case that you vaccinate every neonate"—is *satisfied* but '$\exists xNx \rightarrow !\forall x(Nx \rightarrow Vx)$'—which symbolizes (2) "if there are any neonates, let it be the case that you vaccinate every neonate"—is *avoided*. (By contrast, the *declarative* sentences '$\forall x(Nx \rightarrow Vx)$' and '$\exists xNx \rightarrow \forall x(Nx \rightarrow Vx)$' *are* logically equivalent.) The distinction between (1) and (2) captures a subtle ambiguity in the English sentence (3) "vaccinate every neonate" (cf. Ludwig 1997, 39), an ambiguity that can be revealed by asking: what if there are no neonates? In contexts in which the answer is that the command expressed by (3) is then trivially satisfied, (3) can be paraphrased as (1) and symbolized as '$!\forall x(Nx \rightarrow Vx)$'; but in contexts in which the answer is that the command expressed by (3) is then avoided, (3) can be paraphrased as (2) and symbolized either as '$\exists xNx \rightarrow !\forall x(Nx \rightarrow Vx)$' or, equivalently, as '$\forall x(Nx \rightarrow !Vx)$'.

Similar remarks apply to (4) "vaccinate some neonate": this was symbolized in §2 as '$\exists x(Nx \rightarrow !Vx)$'—which is logically equivalent to '$\exists xNx \rightarrow !\exists x(Nx \,\&\, Vx)$'—but can alternatively be symbolized as '$!\exists x(Nx \,\&\, Vx)$' depending on the context (i.e., depending on whether the command expressed by (4) is taken to be avoided or trivially satisfied if there are no neonates). One might argue that both symbolizations are inadequate: neither of them appears to entail '$\exists xNx$', but one might argue that (4) "vaccinate some neonate" can be paraphrased as (5) "there are neonates; vaccinate at least one of them" and thus appears to entail "there are neonates". In reply, I do not need to take a stand on whether (4) can be paraphrased as (5): if it can, just symbolize (4) in the same way as (5). But how should (5) be symbolized? One might claim that it should be symbolized as '$\exists x(Nx \,\&\, !Vx)$', which is *not* a formula of FOPIL (cf. Clarke 1973, 201; Clarke & Behling 1998, 293; Gensler 1990, 192; 1996, 186; 2002, 185). So one might claim that '$\exists x(Nx \,\&\, !Vx)$' *should* be a formula, and one might propose modifying my definition of a formula by adopting the following additional formation rule: if p is a declarative formula and i is an imperative formula, then $\ulcorner(p \,\&\, i)\urcorner$ and $\ulcorner(i \,\&\, p)\urcorner$ are formulas. Addressing a similar point in §2, I replied in effect that such a rule is unnecessary because, for example, nothing important is lost by symbolizing the two parts of "although you are not going to vaccinate Alice, at least vaccinate Brenda" separately, as '$\sim Va$' and '$!Vb$'. In the present context, however, one might find such a reply unsatisfactory: one might argue that the two parts of (5) "there are neonates; vaccinate at least one of them" cannot be symbolized separately because the second part ("vaccinate at least one of them") "is not by itself a complete imperative [sentence], since it does not contain the referent of the pronoun ['them']" (Castañeda 1963, 228–229). I reply that (5) can be paraphrased as "there are neonates; if there are neonates, vaccinate at least one of them", so the second part of (5) can be symbolized separately as

'$\exists xNx \rightarrow\ !\exists x(Nx\ \&\ Vx)$' (equivalently, as '$\exists x(Nx \rightarrow\ !Vx)$').[11] I conclude that the proposed additional formation rule is unnecessary.[12]

[11] Similar remarks apply to more complex cases; for example, the English sentence "there is only one neonate; vaccinate it" can be paraphrased as "there is only one neonate; if there is only one neonate, vaccinate it", so the second part of the former English sentence can be symbolized separately as '$\forall x((Nx\ \&\ \forall y(Ny \rightarrow x = y)) \rightarrow\ !Vx)$' ("for any *x*, if *x* is the only neonate, vaccinate *x*")—which turns out to be logically equivalent to '$\exists x((Nx\ \&\ \forall y(Ny \rightarrow x = y)) \rightarrow\ !Vx)$' ("for some *x*, if *x* is the only neonate, vaccinate *x*").

[12] One might grant that the proposed rule is unnecessary but might argue that the rule is desirable because it makes *simpler* symbolizations available: symbolizing "there is only one neonate; vaccinate it" as '$\exists x((Nx\ \&\ \forall y(Ny \rightarrow x = y))\ \&\ !Vx)$' would be simpler than symbolizing it, as I propose (see note 11), in terms of both a declarative and an imperative sentence. I reply that adopting the proposed rule would create considerable complications. First, would $\ulcorner(p\ \&\ i)\urcorner$ be (1) a declarative but not an imperative formula, (2) an imperative but not a declarative formula, (3) both a declarative and an imperative formula, or (4) neither a declarative nor an imperative formula? Against (1): if, for example, '$\sim Va\ \&\ !Vb$' is a declarative but not an imperative formula (and sentence), then it seems unavoidable to say that '$\sim Va\ \&\ !Vb$' is true on all and only those interpretations on which '$\sim Va$' is true, and then it seems unavoidable to say that '$\sim Va\ \&\ !Vb$' is logically equivalent to '$\sim Va$'—an absurd result. Against (2): if '$\sim Va\ \&\ !Vb$' is an imperative but not a declarative formula, then it seems unavoidable to say that '$\sim Va\ \&\ !Vb$' is satisfied (or violated) on all and only those interpretations on which '$!Vb$' is satisfied (or violated), and then it seems unavoidable to say that '$\sim Va\ \&\ !Vb$' is logically equivalent to '$!Vb$'—an absurd result. Against (3): if '$\sim Va\ \&\ !Vb$' is both a declarative and an imperative formula, then (assuming that imperative formulas are not true or false and that declarative formulas are not satisfied, violated, or avoided) '$\sim Va\ \&\ !Vb$' is not true or false and is not satisfied, violated, or avoided on any interpretation, and then I do not see what kinds of semantic properties '$\sim Va\ \&\ !Vb$' could have, so I do not see how it could play any non-trivial role in a definition of (semantic) validity. In favor of (4): one could say that '$\sim Va\ \&\ !Vb$' is a *mixed* (i.e., neither a declarative nor an imperative) formula, which is (a) true on all and only those interpretations on which '$\sim Va$' is true and is (b) satisfied (or violated) on all and only those interpretations on which '$!Vb$' is satisfied (or violated). To say that logical equivalence applies to mixed sentences without saying that '$\sim Va\ \&\ !Vb$' is logically equivalent to '$\sim Va$' or to '$!Vb$', one could modify the definition of logical equivalence in §3 as follows: sentences φ and ψ are logically equivalent only if either they are both declarative or they are both imperative *or they are both mixed*, and *mixed* sentences φ and ψ are logically equivalent exactly if, for any interpretation m, φ is true on m exactly if ψ is true on m, φ is satisfied on m exactly if ψ is satisfied on m, and φ is violated on m exactly if ψ is violated on m. One could further say that, if φ is any formula and ψ is a mixed formula, then $\ulcorner(\varphi\ \&\ \psi)\urcorner$ and $\ulcorner(\psi\ \&\ \varphi)\urcorner$ are mixed formulas. To my mind, the complications introduced by this proposal are not worth the benefit of having simpler symbolizations. (Moreover, note that this proposal could not be extended to also recognize '$\sim Va \vee\ !Vb$' as a mixed formula, because then it would seem unavoidable to say that '$\sim Va \vee\ !Vb$' is logically equivalent to '$\sim Va\ \&\ !Vb$'—unless one considerably modified the semantics, as for example Starr 2020 does.) Note finally that even more complications arise if p and i in the proposed formation rule are formulas that are not sentences, so it is unpromising to claim—as Clarke (1973, 201; Clarke & Behling 1998, 293) in effect does—that '$\sim Va\ \&\ !Vb$' is not a formula but '$\exists x(Nx\ \&\ !Vx)$' is nevertheless a formula.

6. Strong and weak derivations

In this section, I introduce a set of inference rules such that a pure imperative argument is weakly valid exactly if its conclusion can be derived from its premises by means of those rules—possibly together with the replacement rules I introduced in §5—and a similar result holds for strong validity.

DEFINITION 4. For any non-empty finite set Γ of imperative sentences and any imperative sentence *i*:

(1) A *strong derivation* of *i* from Γ is a finite sequence of imperative sentences (called the *lines* of the derivation) such that (a) the last line is *i* and (b) each line either is a conjunction of *all* members of Γ or can be obtained from a previous line by applying once either a replacement rule from Table 1 or a pure imperative inference rule (*other than ICE*) from Table 2.

(2) A *weak derivation* of *i* from Γ is a finite sequence of imperative sentences (called the *lines* of the derivation) such that (a) the last line is *i* and (b) each line either is (a member or) a conjunction of members of Γ or can be obtained from a previous line by applying once either a replacement rule from Table 1 or a pure imperative inference rule from Table 2.

Name of rule and abbreviation		**Rule**
Ex Contradictione Quodlibet	ECQ	$!(p \,\&\, {\sim}p) \blacktriangleright i$
Declarative Antecedent Introduction	DAI	$i \blacktriangleright p \rightarrow i$
Imperative Conjunction Elimination	ICE	$i \,\&\, j \blacktriangleright i$

Table 2. Pure imperative inference rules.

In Table 2, for any imperative sentences *i* and *j*, '$i \blacktriangleright j$' abbreviates "from *i*, one can obtain *j*". It follows from Definition 4 that every strong derivation is a weak derivation. Moreover, since replacement rules may be applied in strong derivations, every replacement derivation of an imperative sentence *j* from an imperative sentence *i* is also a strong derivation of *j* from *i* (strictly speaking, from {*i*}). Note two differences between weak and strong derivations. First, all pure imperative inference rules in Table 2 may be applied in a weak derivation, but Imperative Conjunction Elimination (ICE) may *not* be applied in a strong derivation. The motivation behind this difference is that, for example, the argument $\langle\{!Va \,\&\, !Vb\}, !Va\rangle$ is (weakly but) not strongly valid (see Vranas 2011, 411, 416), but strong derivations are intended to correspond to strong validity. Second, any single premise can be the first line of a weak derivation, but no single premise (as opposed to a conjunction of all premises) can be the first line of a strong derivation (unless there is only one premise). The motivation behind this difference is that, for example, the argument $\langle\{!Va, !Vb\}, !Va\rangle$ is (weakly but) not strongly valid (see Vranas 2011, 397).

In the Appendix, I prove that a pure imperative argument is strongly (or weakly) valid exactly if there is a strong (or weak) derivation of its conclusion from the set of its premises. In other words, the natural deduction system that I propose in this paper is *sound and complete*. This is the main result of this paper.

All three rules in Table 2 were also given in Vranas 2019, which proposed a sound and complete natural deduction system for *Sentential* Pure Imperative Logic: interestingly, no new pure imperative inference rules are needed when one introduces quantifiers and identity. Nevertheless, one can show that an imperative analog of classical Universal Instantiation holds in FOPIL. For example, here is a weak derivation of '$Nb \rightarrow !Vb$' from '$\forall x(Nx \rightarrow !Vx)$':

1.	$\forall x(Nx \rightarrow !Vx)$	Premise
2.	$\exists xNx \rightarrow !\forall x(Nx \rightarrow Vx)$	1 Imperative Quantification
3.	$(Nb \vee \exists xNx) \rightarrow !\forall x(Nx \rightarrow Vx)$	2 Declarative Replacement
4.	$(Nb \vee \exists xNx) \rightarrow !((Nb \rightarrow Vb) \& \forall x(Nx \rightarrow Vx))$	3 Declarative Replacement
5.	$(Nb \rightarrow !(Nb \rightarrow Vb)) \& (\exists xNx \rightarrow !\forall x(Nx \rightarrow Vx))$	4 Imperative Conjunction
6.	$Nb \rightarrow !(Nb \rightarrow Vb)$	5 Imperative Conjunction Elimination
7.	$Nb \rightarrow !(Nb \& (Nb \rightarrow Vb))$	6 Absorption
8.	$Nb \rightarrow !(Nb \& Vb)$	7 Declarative Replacement
9.	$Nb \rightarrow !Vb$	8 Absorption

Similarly, an imperative analog of classical Existential Generalization holds in FOPIL: for example, one can show that there is a weak derivation of '$\exists x!Vx$' from '$!Vb$'. Note that the premise '$!Vb$' is an *unconditionally* prescriptive sentence (i.e., a sentence of the form $\ulcorner !p \urcorner$); by contrast, from a *conditionally* prescriptive sentence like '$Nb \rightarrow !Vb$' one cannot derive (at all in FOPIL) '$\exists x(Nx \rightarrow !Vx)$'. This is because existential generalization generalizes disjunctive addition, but imperative disjunctive addition fails in general (although it works for unconditionally prescriptive sentences). Informally: "if you marry, procreate" does not entail "if you marry, procreate, or if you don't marry, procreate"—which is equivalent to "procreate". This is a significant disanalogy between imperative and classical declarative logic.

Appendix: Theorems and proofs

To prove my main result, namely the soundness and completeness of the natural deduction system that I proposed in this paper, I prove first a series of lemmas.

LEMMA 1 (SEMANTIC REPLACEMENT). For any imperative formula i and any formulas φ and ψ, if φ is a subformula of i and $\varphi \Leftrightarrow \psi$ (i.e., φ is logically equivalent to ψ), then $i \Leftrightarrow i(\varphi/\psi)$—where $i(\varphi/\psi)$ is any formula that results from replacing in i at least one occurrence of φ with ψ.

PROOF. The proof is by induction on the number of occurrences of connectives or quantifiers in i. For the base step, take any i in which no connectives or quantifiers occur. Then, for some atomic formula p, i is $\ulcorner !p \urcorner$. So if, for some q, $p \Leftrightarrow q$, then $i(p/q)$, namely $\ulcorner !q \urcorner$, is logically equivalent to $\ulcorner !p \urcorner$: if $u_1, \ldots, u_n$ are the only variables that have at least one free occurrence in p or in q, then for any distinct constants $h_1, \ldots, h_n$ that occur neither in p nor in q, and for any interpretation m, $\ulcorner !p[u_1/h_1, \ldots, u_n/h_n] \urcorner$ is satisfied on m iff $p[u_1/h_1, \ldots, u_n/h_n]$ is true on m (by I1) iff $q[u_1/h_1, \ldots, u_n/h_n]$ is true on m (since $p \Leftrightarrow q$) iff $\ulcorner !q[u_1/h_1, \ldots, u_n/h_n] \urcorner$ is satisfied on m (and similarly for violation). For the inductive step, take any natural number n and suppose (*induction hypothesis*) that, for any i with at most n occurrences of connectives or quantifiers, and any φ and ψ such that φ is a subformula of i and $\varphi \Leftrightarrow \psi$, $i \Leftrightarrow i(\varphi/\psi)$. To complete the proof, take any i with at most $n + 1$ occurrences of connectives or quantifiers and any φ and ψ such that φ is a *proper* subformula of i (the case in which φ is i is trivial) and $\varphi \Leftrightarrow \psi$. To prove that $i \Leftrightarrow i(\varphi/\psi)$, there are ten cases to consider, depending on whether i is $\ulcorner !p \urcorner$, $\ulcorner \sim j \urcorner$, $\ulcorner j \,\&\, k \urcorner$, $\ulcorner j \vee k \urcorner$, $\ulcorner p \rightarrow j \urcorner$, $\ulcorner j \rightarrow p \urcorner$, $\ulcorner p \leftrightarrow j \urcorner$, $\ulcorner j \leftrightarrow p \urcorner$, $\ulcorner \forall uj \urcorner$, or $\ulcorner \exists uj \urcorner$. For the first eight cases, the proof parallels the proof of Theorem 3.1 in Vranas 2019, so consider the last two cases: suppose i is $\ulcorner \forall uj \urcorner$ or $\ulcorner \exists uj \urcorner$. Then φ is a subformula of j, and $i(\varphi/\psi)$ is $\ulcorner \forall uj' \urcorner$ or $\ulcorner \exists uj' \urcorner$, where j' is $j(\varphi/\psi)$. By the induction hypothesis, $j \Leftrightarrow j'$. Suppose $u_1, \ldots, u_n$ are the only variables that have at least one free occurrence in i or in $i(\varphi/\psi)$, and take any distinct constants $h_1, \ldots, h_n$ that occur neither in i nor in $i(\varphi/\psi)$. Let k be $j[u_1/h_1, \ldots, u_n/h_n]$, and let k' be $j'[u_1/h_1, \ldots, u_n/h_n]$. Then $i \Leftrightarrow i(\varphi/\psi)$ iff (1) $\ulcorner \forall uk \urcorner \Leftrightarrow \ulcorner \forall uk' \urcorner$. But (1) holds (by I9 and I10) because, for any interpretation m and any member o of the domain of m, o satisfies k on m iff o satisfies k' on m (and similarly for violating and avoiding): if o satisfies k on m, then, for any constant h that occurs neither in k nor in k', $k[u/h]$ is satisfied on $m[h/o]$, so $k'[u/h]$ is also satisfied on $m[h/o]$ (since $j \Leftrightarrow j'$, $k[u/h]$ is $j[u/h, u_1/h_1, \ldots, u_n/h_n]$, and $k'[u/h]$ is $j'[u/h, u_1/h_1, \ldots, u_n/h_n]$), and thus o satisfies k' on m (and, similarly, vice versa).

LEMMA 2 (CANONICAL FORM). For any imperative formula i, there are declarative formulas p and q such that $i \dashv\vdash \ulcorner p \rightarrow\ !q \urcorner$ (i.e., there is a replacement derivation of $\ulcorner p \rightarrow\ !q \urcorner$ from i).

PROOF. The proof is by induction on the number of occurrences of connectives or quantifiers in i. For the base step, take any i in which no connectives or quantifiers occur. Then, for some atomic formula p, i is $\ulcorner !p \urcorner$, and then, by TA, $i \dashv\vdash \ulcorner (p \vee \sim p) \rightarrow\ !p \urcorner$. For the inductive step, take any natural number n and suppose (*induction hypothesis*) that, for any i with at most n occurrences of connectives or quantifiers, there are p and q such that $i \dashv\vdash \ulcorner p \rightarrow\ !q \urcorner$. To complete the proof, take any i with at most $n + 1$ occurrences of connectives or quantifiers. There are the same ten cases to consider as in the proof of Lemma 1. For the first eight cases, the proof parallels the proof of Theorem 4.4 in Vranas 2019, so consider the last two cases: suppose i is $\ulcorner \forall uj \urcorner$ or $\ulcorner \exists uj \urcorner$. Then, by the induction hypothesis, $j \dashv\vdash \ulcorner p \rightarrow\ !q \urcorner$, and then, by IQ, $i \dashv\vdash \ulcorner \exists up \rightarrow\ !\forall u(p \rightarrow q) \urcorner$ or $i \dashv\vdash \ulcorner \exists up \rightarrow\ !\exists u(p\ \&\ q) \urcorner$.

LEMMA 3 (SOUNDNESS OF REPLACEMENT RULES). For any imperative formulas i and j, if $i \bowtie j$ according to a replacement rule in Table 1, then $i \Leftrightarrow j$.

PROOF. I examine only IQ: for the soundness of the remaining replacement rules, see the proof of Theorem 4.2 in Vranas 2019. Let i be $\ulcorner \exists u(p \rightarrow\ !q) \urcorner$, and let j be $\ulcorner \exists up \rightarrow\ !\exists u(p\ \&\ q) \urcorner$. Suppose $u_1, \ldots, u_n$ are the only variables that have at least one free occurrence in i or in j, and take any distinct constants $h_1, \ldots, h_n$ that occur neither in i nor in j. Let p' be $p[u_1/h_1, \ldots, u_n/h_n]$, and let q' be $q[u_1/h_1, \ldots, u_n/h_n]$. Let i' be $\ulcorner \exists u(p' \rightarrow\ !q') \urcorner$, and let j' be $\ulcorner \exists up' \rightarrow\ !\exists u(p'\ \&\ q') \urcorner$. Then $i \Leftrightarrow j$ iff (1) $i' \Leftrightarrow j'$. But (1) holds because, for any interpretation m: by I5 and I1, j' is satisfied on m iff both $\ulcorner \exists up' \urcorner$ and $\ulcorner \exists u(p'\ \&\ q') \urcorner$ are true on m; i.e., by D11, iff (2) some member of the domain Δ of m verifies $\ulcorner p'\ \&\ q' \urcorner$ on m (and (3) some member of Δ verifies p' on m, but (3) follows from (2)); i.e., iff, for some member o of Δ and any constant h that does not occur in $\ulcorner p'\ \&\ q' \urcorner$, $\ulcorner p'[u/h]\ \&\ q'[u/h] \urcorner$ is true—equivalently: $\ulcorner p'[u/h] \rightarrow\ !q'[u/h] \urcorner$ is satisfied—on $m[h/o]$; i.e., iff some member of Δ satisfies $\ulcorner p' \rightarrow\ !q' \urcorner$ on m; i.e., by I10, iff i' is satisfied on m (and similarly for violation on m). One can similarly show that $\ulcorner \forall u(p \rightarrow\ !q) \urcorner \Leftrightarrow \ulcorner \exists up \rightarrow\ !\forall u(p \rightarrow q) \urcorner$.

THEOREM 1 (SOUNDNESS AND COMPLETENESS FOR REPLACEMENT DERIVATIONS). For any imperative formulas i and j, $i \Leftrightarrow j$ if (*soundness*) and only if (*completeness*) $i \dashv\vdash j$.

PROOF OF SOUNDNESS. Suppose $i \dashv\vdash j$. The proof is by induction on the number of lines of a replacement derivation. For the base step, suppose there is a one-line replacement derivation of j from i. Then i is the same formula as j and thus $i \Leftrightarrow j$. For the inductive step, take any non-zero natural number

n and suppose (*induction hypothesis*) that, if there is a replacement derivation with n lines of j from i, then $i \Leftrightarrow j$. To complete the proof, take any replacement derivation with $n + 1$ lines of j from i. Then j can be obtained from the n-th line k by applying once a replacement rule, so j is $k(\varphi/\psi)$, where φ is a subformula of k and ψ is a formula such that $\varphi \bowtie \psi$. By the induction hypothesis, (1) $i \Leftrightarrow k$. By Lemma 3, $\varphi \Leftrightarrow \psi$ if φ and ψ are imperative formulas; if they are declarative, then j can be obtained from k by applying once DR, so $\varphi \dashv\vdash_{CFOL} \psi$ and thus again $\varphi \Leftrightarrow \psi$. By Lemma 1, $k \Leftrightarrow k(\varphi/\psi)$; i.e., (2) $k \Leftrightarrow j$. By (1) and (2), $i \Leftrightarrow j$.

<u>Proof of Completeness</u>. Suppose $i \Leftrightarrow j$. By Lemma 2, (1) $i \dashv\vdash \ulcorner p \rightarrow !q \urcorner$ and thus (by soundness) $i \Leftrightarrow \ulcorner p \rightarrow !q \urcorner$, and (2) $j \dashv\vdash \ulcorner p' \rightarrow !q' \urcorner$ and thus $j \Leftrightarrow \ulcorner p' \rightarrow !q' \urcorner$. Then (3) $\ulcorner p \rightarrow !q \urcorner \Leftrightarrow \ulcorner p' \rightarrow !q' \urcorner$. Then one can show (see the proof of Theorem 4.5 in Vranas 2019) that $p \Leftrightarrow p'$, so (4) $p \dashv\vdash_{CFOL} p'$, and that $\ulcorner p \,\&\, q \urcorner \Leftrightarrow \ulcorner p' \,\&\, q' \urcorner$, so (5) $\ulcorner p \,\&\, q \urcorner \dashv\vdash_{CFOL} \ulcorner p' \,\&\, q' \urcorner$. Then: $i \dashv\vdash \ulcorner p \rightarrow !q \urcorner$ (by (1)), so $i \dashv\vdash \ulcorner p \rightarrow !(p \,\&\, q) \urcorner$ (by AB), so $i \dashv\vdash \ulcorner p' \rightarrow !(p \,\&\, q) \urcorner$ (by (4) and DR), so $i \dashv\vdash \ulcorner p' \rightarrow !(p' \,\&\, q') \urcorner$ (by (5) and DR), so $i \dashv\vdash \ulcorner p' \rightarrow !q' \urcorner$ (by AB), so finally $i \dashv\vdash j$ (by (2)).

<u>Corollary 1 (of Lemma 2 and Theorem 1)</u>. For any imperative formula i, there are declarative formulas p and q such that $i \Leftrightarrow \ulcorner p \rightarrow !q \urcorner$.

<u>Lemma 4 (Semantic Equivalence)</u>. For any imperative sentences i and j: (1) i strongly entails j iff either i is a contradiction or both (a) i is satisfied on every interpretation on which j is satisfied and (b) i is violated on every interpretation on which j is violated; (2) i weakly entails j iff both (a) j is avoided on every interpretation on which i is avoided and (b) i is violated on every interpretation on which j is violated.

I omit the proof, because it parallels the proof of Theorem 5.3 in Vranas 2019. Lemma 4 shows that Definition 2, which quantifies over declarative sentences, is equivalent to a definition of (strong and weak) validity that does *not* quantify over declarative sentences (cf. Vranas 2011: 394). It follows from Lemma 4 that i strongly entails j only if i weakly entails j. Moreover, Lemma 4 has the following consequence (for a proof, see the proof of Theorem 6.2 in Vranas 2019):

<u>Corollary 2 (Soundness of Inference Rules)</u>. For any declarative sentence p and any imperative sentences i and j: (1) $\ulcorner !(p \,\&\, {\sim}p) \urcorner$ strongly (and thus weakly) entails i; (2) i strongly (and thus weakly) entails $\ulcorner p \rightarrow i \urcorner$; (3) $\ulcorner i \,\&\, j \urcorner$ weakly entails i.

<u>Theorem 2 (Soundness and Completeness for Strong and Weak Derivations)</u>. A pure imperative argument $\langle \Gamma, i \rangle$ is (1) *strongly* valid if

(*soundness*) and only if (*completeness*) there is a *strong* derivation of i from Γ, and is (2) *weakly* valid if (*soundness*) and only if (*completeness*) there is a *weak* derivation of i from Γ.

PROOF OF SOUNDNESS. The proof is by induction on the number of lines of a strong or weak derivation. For the base step, suppose there is a one-line strong (case 1) or weak (case 2) derivation of i from Γ. In case 1, i is a conjunction of all members of Γ and thus (by Definition 2) Γ strongly entails i. In case 2, i is (a member or) a conjunction of members of Γ; so, if i is not a conjunction of *all* members of Γ (if it is, the proof proceeds as in case 1), there is a conjunction j of the remaining members of Γ, and $\ulcorner i \,\&\, j \urcorner$ is a conjunction of all members of Γ. Then Γ weakly entails i because, by Definition 2, Γ weakly entails $\ulcorner i \,\&\, j \urcorner$, and by Corollary 2, $\ulcorner i \,\&\, j \urcorner$ weakly entails i. For the inductive step, take any non-zero natural number n and suppose (*induction hypothesis*) that: (case 1) if there is a strong derivation with at most n lines of i from Γ, then Γ strongly entails i; (case 2) if there is a weak derivation with at most n lines of i from Γ, then Γ weakly entails i. To complete the proof, take any strong (case 1) or weak (case 2) derivation with at most $n + 1$ lines of i from Γ. Suppose that i is *not* a conjunction of all (case 1) or some (case 2) members of Γ (if it is, the proof proceeds as in the base step). Then i can be obtained from an n'-th line j ($n' \leq n$) by applying once (case 1) ECQ, DAI, or a replacement rule, or (case 2) any inference or replacement rule. Then (1_s) j strongly entails i in case 1 and (1_w) j weakly entails i in case 2 (by Corollary 2). By the induction hypothesis and the fact that the sequence of the first n' lines of the strong (case 1) or weak (case 2) derivation of i from Γ is a strong (case 1) or weak (case 2) derivation with at most n lines of j from Γ, (2_s) Γ strongly entails j in case 1, and (2_w) Γ weakly entails j in case 2. By (1_s) and (2_s), Γ strongly entails i in case 1. Similarly, by (1_w) and (2_w), Γ weakly entails i in case 2.

PROOF OF COMPLETENESS. Take any pure imperative argument $\langle \Gamma, i \rangle$ and any conjunction i' of all members of Γ. By Lemma 2, (1) $i \dashv\vdash \ulcorner p \rightarrow\, !q \urcorner$ and (2) $i' \dashv\vdash \ulcorner p' \rightarrow\, !q' \urcorner$. *Case 1*: Γ strongly entails i. Then (3) i' strongly entails i (by Definition 2). Case 1a: i' is a contradiction. Then, for any r, $i' \Leftrightarrow \ulcorner !(r \,\&\, {\sim}r) \urcorner$ and thus (by Theorem 1) there is a replacement—and thus a strong—derivation of $\ulcorner !(r \,\&\, {\sim}r) \urcorner$ from i'. Then there is a strong derivation of i from i' (and thus from Γ), since i can be obtained from $\ulcorner !(r \,\&\, {\sim}r) \urcorner$ by ECQ. Case 1b: i' is not a contradiction. Then, by (3) and Lemma 4, i' is satisfied on every interpretation on which i is satisfied, and i' is violated on every interpretation on which i is violated. It follows, by CFOL, that (4) $p \dashv\vdash_{CFOL} \ulcorner p \,\&\, p' \urcorner$ and (5) $\ulcorner p \,\&\, q' \urcorner \dashv\vdash_{CFOL} \ulcorner p \,\&\, q \urcorner$. To conclude: there is a strong derivation from Γ of i' (by Definition 4), and thus of $\ulcorner p' \rightarrow\, !q' \urcorner$ (by (2)), and thus of $\ulcorner p \rightarrow (p' \rightarrow\, !q') \urcorner$ (by DAI), and thus of $\ulcorner (p \,\&\, p') \rightarrow\, !q' \urcorner$ (by EX), and thus of $\ulcorner p \rightarrow\, !q' \urcorner$ (by (4) and DR), and thus of $\ulcorner p \rightarrow\, !(p \,\&$

$q')^\urcorner$ (by AB), and thus of $^\ulcorner p \rightarrow !(p \;\&\; q)^\urcorner$ (by (5) and DR), and thus of $^\ulcorner p \rightarrow !q^\urcorner$ (by AB), and thus finally of i (by (1)). *Case 2*: Γ weakly entails i. Then i' weakly entails i (by Definition 4 and the observation that any member or conjunction of members of Γ can be obtained from i' by applying replacement rules or ICE or both). Then, by Lemma 4, i is avoided on every interpretation on which i' is avoided, and i' is violated on every interpretation on which i is violated. It follows, by CFOL, that (4) holds and also (6) $^\ulcorner p \;\&\; q^\urcorner \dashv\vdash_{CFOL} {}^\ulcorner(q \;\&\; (p \;\&\; q')^\urcorner$. To conclude: there is a weak derivation from Γ of $^\ulcorner p \rightarrow !(p \;\&\; q')^\urcorner$ (by (2), DAI, EX, (4) and DR, and AB, as in case 1b), and thus of $^\ulcorner p \rightarrow !(q \;\&\; (p \;\&\; q'))^\urcorner$ (by (6) and DR), and thus of $^\ulcorner p \rightarrow !(p \;\&\; ((p \rightarrow q) \;\&\; (p \rightarrow q')))^\urcorner$ (by CFOL and DR), and thus of $^\ulcorner p \rightarrow !((p \rightarrow q) \;\&\; (p \rightarrow q'))^\urcorner$ (by AB), and thus of $^\ulcorner(p \vee p) \rightarrow !((p \rightarrow q) \;\&\; (p \rightarrow q'))^\urcorner$ (by CFOL and DR), and thus of $^\ulcorner(p \rightarrow !q) \;\&\; (p \rightarrow !q')^\urcorner$ (by IC), and thus of $^\ulcorner p \rightarrow !q^\urcorner$ (by ICE), and thus finally of i (by (1)).

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