

NEW FOUNDATIONS FOR IMPERATIVE LOGIC IV: NATURAL DEDUCTION*

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Abstract. I present sound and complete natural deduction systems for (quantified modal) imperative logic, in five steps. (1) Syntax: I introduce *imperative formal languages* by using the imperative operator ‘!’ (“let it be the case that”); e.g., if ‘ A ’ is a declarative sentence, ‘! A ’ is an imperative sentence. (2) Semantics: I introduce *interpretations* of imperative formal languages, and (building on previous work) I define what it is for a declarative sentence to *sustain* a (declarative or imperative) sentence on an interpretation. (3) Semantic validity: I define an argument to be *semantically valid* exactly if, on every interpretation, every declarative sentence that sustains its premises also sustains its conclusion. (4) Syntactic validity: I define an argument to be *syntactically valid* exactly if its conclusion can be derived from its premises by applying certain “natural” replacement and inference rules that I introduce. (5) Soundness and completeness: I prove that semantic and syntactic validity coincide.

1. Introduction

Here is a logic test for you. Symbolize sentences (1)-(3) below by using the provided symbols, and then show by natural deduction that the corresponding argument is valid:

Advisor: (1) Take only courses in moral or political philosophy this term. (You have already satisfied all other requirements.)

Student: (2) No courses in political philosophy can be offered this term. (The only person who taught such courses has retired.)

Advisor: Then (3) take only courses in moral philosophy this term.

(Mx : x is a course in moral philosophy offered this term; Px : x is a course in political philosophy offered this term; Tx : the student takes x this term.)

You may complain that this test is unfair: you were never taught how to symbolize *imperative* English sentences (like “take only courses in moral philosophy this term”) or how natural deduction applies to arguments with imperative premises or conclusions. The complaint is reasonable: almost no logic textbook covers these topics. This is hardly surprising, given that almost no technical literature covers these topics either.¹ In this paper, I take steps to remedy this situation. I present sound and complete natural deduction systems for imperative logic.

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¹ To my knowledge, only two logic textbooks cover symbolization of imperative English sentences and natural deduction for imperative logic: Clarke & Behling 1998 (a descendant of Clarke 1973) and Gensler 2002 (a descendant of Gensler 1990; see also Gensler 1996: 181-6). These textbooks, however, rely on inadequate definitions of validity for imperative arguments (see Vranas 2011, 2016). Relying on my definition of validity for the

I examine four logics. I begin (in §2) with *sentential pure imperative logic* (SPIL), which does not include quantifiers or modal operators and deals with arguments from imperative premises to imperative conclusions. I continue (in §3) with *sentential modal imperative logic* (SMIL), which includes modal operators and deals with arguments from declarative or imperative premises (or both) to declarative or imperative conclusions. I end (in §4 and §5) with *quantified pure imperative logic* (QPIL) and *quantified modal imperative logic* (QMIL), which include quantifiers and identity (but no function symbols). For each logic, I provide an imperative formal language as well as replacement and inference rules that can be used to derive a conclusion from a set of premises. The replacement and inference rules are intended to represent “natural” patterns of reasoning, but their justification is not limited to intuitions about “naturalness”. The justification relies crucially on the result—which I prove in the Appendix—that derivability by those rules corresponds to a semantic definition of argument validity that I have developed at length in previous papers (Vranas 2011, 2016; see also Vranas 2008, 2010, 2013) and that I develop further here by introducing *interpretations* of imperative formal languages. I do not presuppose any familiarity with the previous papers: I repeat and often briefly motivate here the relevant previous results.

2. Sentential pure imperative logic (SPIL)

2.1. Syntax

The (imperative formal) language of SPIL has the following symbols: the connectives ‘ \sim ’, ‘ $\&$ ’, ‘ \vee ’, ‘ \rightarrow ’, and ‘ \leftrightarrow ’, the punctuation symbols ‘(’ and ‘)’, the *imperative operator* ‘!’ (“let it be the case that”), and the (infinitely many) sentence letters ‘ A ’, ‘ B ’, ..., ‘ Z ’, ‘ A' ’, ‘ B' ’, ..., ‘ Z' ’, ‘ A'' ’, ‘ B'' ’, (One could also define languages of SPIL with different sentence letters or with only finitely many sentence letters, but for simplicity I define only a single language of SPIL.) The *declarative sentences* of (the language of) SPIL are all and only those finite strings of symbols (understood as ordered n -tuples of symbols) that either are sentence letters or can be built up from sentence letters by applying at least once the following formation rule: if p and q are declarative sentences, then $\lceil \sim p \rceil$, $\lceil p \& q \rceil$, $\lceil p \vee q \rceil$, $\lceil p \rightarrow q \rceil$, and $\lceil p \leftrightarrow q \rceil$ are also declarative sentences. The *imperative sentences* of (the language of) SPIL are all and only those finite strings of symbols that can be built up from declarative sentences by applying the following formation rules (R1 must be applied at least once):

- (R1) If p is a declarative sentence, then $\lceil !p \rceil$ is an imperative sentence.
- (R2) If i and j are imperative sentences, then $\lceil \sim i \rceil$, $\lceil i \& j \rceil$, and $\lceil i \vee j \rceil$ are also imperative sentences.
- (R3) If p is a declarative sentence and i is an imperative sentence, then $\lceil p \rightarrow i \rceil$, $\lceil i \rightarrow p \rceil$, $\lceil p \leftrightarrow i \rceil$, and $\lceil i \leftrightarrow p \rceil$ are imperative sentences.

A *sentence* (of SPIL) is either a declarative sentence or an imperative sentence. It follows from these definitions that a sentence is imperative exactly if it contains at least one occurrence of ‘!’ and is declarative exactly if it contains no occurrence of ‘!’ (so no sentence is both declarative and imperative). For simplicity, I usually omit outermost parentheses, writing for example ‘ $M \rightarrow (!P \rightarrow C)$ ’ instead of ‘ $(M \rightarrow (!P \rightarrow C))$ ’. Here are some examples of how imperative English sen-

special case of arguments with only imperative premises and conclusions (Vranas 2011), Hansen (2014) has provided sound and complete sets of inference rules for a formal language with only one imperative connective and without quantifiers or modal operators.

tences can be symbolized in (the language of) SPIL (P stands for “you procreate”, M for “you marry”, and so on):

Procreate: $!P$

Don't procreate: $\sim!P$

Marry and procreate: $!M \& !P$

If you marry, procreate: $M \rightarrow !P$

Procreate only if you marry: $!P \rightarrow M$

Procreate if and only if you marry: $!P \leftrightarrow M$

Don't procreate without marrying: $\sim!(P \& \sim M)$

If you marry, then procreate only if you copulate: $M \rightarrow (!P \rightarrow C)$

If you marry and ovulate, then copulate and procreate: $(M \& O) \rightarrow (!C \& !P)$

Neither procreate if you don't marry nor copulate if you don't ovulate: $\sim(\sim M \rightarrow !P) \& \sim(\sim O \rightarrow !C)$

One might wonder why “marry and procreate” was symbolized as $!M \& !P$ instead of $!(M \& P)$. I reply that either symbolization will do: (1) $!M \& !P$ symbolizes “let it be the case that you marry, and let it be the case that you procreate”, (2) $!(M \& P)$ symbolizes “let it be the case that you both marry and procreate”, (3) both English sentences are adequate paraphrases of “marry and procreate”,² and (4) it turns out (see §2.2) that $!M \& !P$ and $!(M \& P)$ are logically equivalent. Similarly, it turns out that $\sim!P$ (“let it not be the case that you procreate”) and $!\sim P$ (“let it be the case that you don't procreate”) are logically equivalent.³ Note that $\sim!P$ is a *negation*, namely a sentence of the form $\lceil \sim\phi \rceil$ (where ϕ is a declarative or imperative sentence), but $!\sim P$ is what I call an *unconditionally prescriptive sentence*, namely a sentence of the form $\lceil !p \rceil$ (where p is a declarative sentence).

One might wonder why I use a single set of connectives both for declarative and for imperative logical operations; why not use instead, for example, ‘&’ for declarative conjunction and ‘&_i’ for imperative conjunction? I reply that the proliferation of connectives would make the notation cumbersome. Note that English likewise uses a single set of coordinating conjunctions both for declarative and for imperative (syndetic) coordination, as the two occurrences of “and” in “if you marry and ovulate, then copulate and procreate” illustrate. One might argue that this ambiguity is an undesirable feature of English and should be eliminated in a formal language. I reply that my use of a single set of connectives does not result in any confusion: it is always clear whether (for example) the ampersand is connecting declarative or imperative sentences, and semantically (see §2.2) the ampersand is treated differently in the two cases.

² Changing slightly the example, one might argue that (1) “let it be the case that Pat procreates” is not an adequate paraphrase of (2) “Pat, procreate” because (2) is addressed to Pat but (1) is not. In reply, consider the following parallel argument concerning *declarative* sentences: (3) “I predict that Pat will procreate” is not an adequate paraphrase of (4) “Pat, I predict that you will procreate” because (4) is addressed to Pat but (3) is not. This argument fails: (3) is an adequate paraphrase of (4), in the sense that (3) and (4) normally express the same *proposition*. Similarly, (1) is an adequate paraphrase of (2), in the sense that—in my preferred terminology—(1) and (2) normally express the same *prescription* (as I argue in Vranas 2008: 554 n. 14, n. 15).

³ Cf. Parsons 2013: 84-5; contrast Clark-Younger & Girard 2013. Imperative English sentences have *imperative negations* (which are also imperative English sentences; e.g., an imperative negation of “pay” is “don't pay”), but arguably also have *permissive negations* (which are permissive English sentences; e.g., a permissive negation of “pay” is “you may fail to pay”). I do not deal with permissive sentences in this paper; this is a topic for future research.

According to the first formation rule for imperative sentences (namely R1), if p is *any* declarative sentence, then $\lceil !p \rceil$ is an imperative sentence. This is as it should be, because prefixing *any* declarative English sentence with “let it be the case that” yields an imperative English sentence. For example, “let it be the case that last week I died” is an imperative English sentence, even if one that would hardly ever be used (cf. Vranas 2008: 555 n. 17).⁴ By contrast, certain strings of symbols of SPIL are *not* sentences. (1) If i is an imperative sentence, then $\lceil !i \rceil$ is *not* a sentence. For example, $\lceil !M \rceil$ is not a sentence. This is as it should be, because “let it be the case that let it be the case that you marry” is not an English sentence; more generally, prefixing an *imperative* English sentence with “let it be the case that” does not yield an English sentence.⁵ (2) If i and j are imperative sentences, then $\lceil (i \rightarrow j) \rceil$ and $\lceil (i \leftrightarrow j) \rceil$ are *not* sentences (contrast Castañeda 1975: 113-5). For example, $\lceil (!M \rightarrow !P) \rceil$ and $\lceil (!M \leftrightarrow !P) \rceil$ are not sentences. This is as it should be, because “if marry, procreate” (or “marry only if procreate”) and “if and only if marry, procreate” (or “marry if and only if procreate”) are not English sentences. (3) If p is a declarative sentence and i is an imperative sentence, then $\lceil (p \& i) \rceil$, $\lceil (i \& p) \rceil$, $\lceil (p \vee i) \rceil$, and $\lceil (i \vee p) \rceil$ are *not* sentences (cf. Vranas 2008: 560 n. 41; contrast Fox 2012: 885-6). For example, $\lceil (\sim M \& !P) \rceil$ is not a sentence. This may seem undesirable, because “you are not going to marry, but nevertheless procreate” *is* an English sentence. I reply that nothing important is lost by symbolizing the two parts of the English sentence separately, as $\lceil \sim M \rceil$ and $\lceil !P \rceil$. Counting $\lceil (\sim M \& !P) \rceil$ as a sentence would complicate the syntax without yielding any commensurate benefit.⁶

⁴ In case one thinks that R1 is too permissive (because English sentences like “let it be the case that last week I died” and “let it be the case that the Earth revolves around the Sun” should be excluded from consideration; see, e.g., Rescher 1966: 34), one can (1) designate some sentence letters as *agential and future-directed* sentence letters, (2) designate those declarative sentences that contain only such sentence letters as *agential and future-directed* declarative sentences, and (3) replace R1 with: (R1') if p is an agential and future-directed declarative sentence, then $\lceil !p \rceil$ is an imperative sentence.

⁵ One might ask: why should English be the adjudicator of a formal language? I answer that I am interested in a formal language that corresponds to the way we use imperative (and declarative) sentences in English (and other natural languages). One might claim that (1) “let it be the case that you (will) let it be the case that you marry” *is* an English sentence, and one might take this as a reason to count $\lceil !M \rceil$ as a sentence logically equivalent to $\lceil !M \rceil$ (cf. Chellas 1971: 124-5). I agree that (1) is an English sentence, but this is not a reason to count $\lceil !M \rceil$ as a sentence, because “let it be the case that”, understood *impersonally*, does *not* occur twice in (1): only the first occurrence of “let” in (1) is impersonal. (Note that “you (will) let it be the case that you marry”—in contrast to “let it be the case that you marry”—is a *declarative* English sentence.)

⁶ See §4.1 (especially note 25) for further discussion of “mixed” English sentences like “you are not going to marry, but nevertheless procreate”. To my knowledge, the imperative operator was introduced into formal languages by Mally (1926; see also Hofstadter & McKinsey 1939). Two alternatives to R1 have been proposed in the literature. (1) Clarke and Behling (1998: 282-4; cf. Clarke 1973: 192-3) propose *postfixing* (instead of prefixing) sentence letters with ‘!’ to form imperative sentences. To allow for example $\lceil (A \rightarrow B) \rceil$ —in addition to $\lceil A \rightarrow B \rceil$ —to count as a sentence, generalize the proposal as follows: postfixing *any* declarative sentence (not just sentence letters) with ‘!’ yields an imperative sentence. But then is $\lceil \sim A \rceil$ a negation or an unconditionally prescriptive sentence? Resolving the ambiguity by saying that, if ϕ is a sentence, then $\lceil (\sim\phi) \rceil$ (instead of $\lceil \sim\phi \rceil$) is a sentence (so that $\lceil \sim A \rceil$ —i.e., $\lceil (\sim A) \rceil$ with the outermost parentheses omitted—is a negation but $\lceil (\sim A) \rceil$ is an unconditionally prescriptive sentence) results in a proliferation of parentheses, as for example in $\lceil (\sim A) \& (\sim B) \rceil$ —which corresponds in my notation to $\lceil !\sim A \& !\sim B \rceil$. Moreover, the proposal introduces an unwelcome asymmetry between modal (and other sentential) operators, which are prefixed, and the imperative operator, which is postfixed. (2) Gensler (1990: 190, 1996: 182, 2002: 184) proposes *underlining* sentence letters (instead of prefixing them with ‘!’) to form imperative sentences. To allow for example $\lceil \underline{A \rightarrow B} \rceil$ —in addition to $\lceil A \rightarrow B \rceil$ —to count as a sentence, generalize the proposal as follows: underlining *any* declarative sentence (not just sentence letters) yields an imperative sentence. This proposal results in a proliferation of underlining, as for example in $\lceil \underline{(A \& (B \vee C))} \rightarrow D \rceil$ —which corresponds in my

2.2. Semantics

An *interpretation* of the language of SPIL is an ordered pair whose first coordinate is a set of sentence letters (namely—see below—those sentence letters that are true on the interpretation) and whose second coordinate is a *favoring relation* (namely—see below—a three-place relation on declarative sentences that satisfies certain conditions). Declarative sentences are *true* or *false* on interpretations, and imperative sentences are *satisfied*, *violated*, or *avoided* on interpretations. Specifically, for any declarative sentences p and q , any imperative sentences i and j , and any interpretation m :

Truth and falsity of a declarative sentence on an interpretation

- (C1) A sentence letter is true on m iff (i.e., exactly if) it is a member of the first coordinate of m .
- (C2) $\lceil \sim p \rceil$ is true on m iff p is not true on m .
- (C3) $\lceil p \ \& \ q \rceil$ is true on m iff both p and q are true on m .
- (C4) $\lceil p \vee q \rceil$ is true on m iff $\lceil \sim(\sim p \ \& \ \sim q) \rceil$ is true on m .
- (C5) $\lceil p \rightarrow q \rceil$ is true on m iff $\lceil \sim p \vee q \rceil$ is true on m .
- (C6) $\lceil p \leftrightarrow q \rceil$ is true on m iff $\lceil (p \rightarrow q) \ \& \ (q \rightarrow p) \rceil$ is true on m .
- (C7) p is false on m iff p is not true on m .

Satisfaction, violation, and avoidance of an imperative sentence on an interpretation

- (C8) $\lceil !p \rceil$ is (a) satisfied on m iff p is true on m , and is (b) violated on m iff p is false on m . (Informally: “procreate” is satisfied iff you procreate, and is violated iff you do not procreate.)
- (C9) $\lceil \sim i \rceil$ is (a) satisfied on m iff i is violated on m , and is (b) violated on m iff i is satisfied on m . (Informally: “don’t procreate” is satisfied iff you do not procreate, namely iff “procreate” is violated, and is violated iff you procreate, namely iff “procreate” is satisfied.)
- (C10) $\lceil i \ \& \ j \rceil$ is (a) satisfied on m iff either (i) both i and j are satisfied on m or (ii) one of i and j is satisfied on m and the other one is neither satisfied nor violated on m , and is (b) violated on m iff at least one of i and j is violated on m . (Note that $\lceil i \ \& \ j \rceil$ can be satisfied on m even if not both i and j are satisfied on m (disjunct (ii) in (a)). Informally: “if you marry, procreate, and if you don’t marry, procreate” is equivalent to “procreate (regardless of whether you marry)”, so it is satisfied if you procreate without marrying and thus if “if you marry, procreate” is not satisfied (see C12 below).)
- (C11) $\lceil i \vee j \rceil$ is satisfied (or violated) on m iff $\lceil \sim(\sim i \ \& \ \sim j) \rceil$ is satisfied (or violated) on m . (Informally: “marry or procreate” is equivalent to the negation of “neither marry nor procreate”.)
- (C12) $\lceil p \rightarrow i \rceil$ is (a) satisfied on m iff both p is true on m and i is satisfied on m , and is (b) violated on m iff both p is true on m and i is violated on m . (Informally: “if you marry, procreate” is satisfied iff you both marry and procreate, and is violated iff you marry without procreating.)
- (C13) $\lceil i \rightarrow p \rceil$ is satisfied (or violated) on m iff $\lceil \sim p \rightarrow \sim i \rceil$ is satisfied (or violated) on m . (Informally: “procreate only if you marry” is equivalent to “if you don’t marry, don’t procreate”.)
- (C14) $\lceil p \leftrightarrow i \rceil$ is satisfied (or violated) on m iff $\lceil (p \rightarrow i) \ \& \ (i \rightarrow p) \rceil$ is satisfied (or violated) on m and also iff $\lceil i \leftrightarrow p \rceil$ is satisfied (or violated) on m . (Informally: “if and only if you marry, procreate” is equivalent to “if you marry, procreate, and procreate only if you marry” and is also equivalent to “procreate if and only if you marry”.)

notation to $\lceil !(A \ \& \ (B \vee C)) \rightarrow D \rceil$. Moreover, the proposal results in a proliferation of symbols: if sentences are ordered n -tuples of symbols and $\lceil A \rightarrow B \rceil$ is a sentence, then $\lceil A \rceil$, $\lceil \rightarrow \rceil$, and $\lceil B \rceil$ are symbols.

(C15) i is avoided on m iff i is neither satisfied nor violated on m . (Informally: “procreate” is avoided iff you neither procreate nor do not procreate, namely never, and “if you marry, procreate” is avoided iff you neither both marry and procreate nor marry without procreating, namely iff you do not marry.)

See Vranas 2008: 532-45 for a detailed defense of C8-C15. A *tautology* is either a declarative sentence that is true on every interpretation (a *declarative tautology*; e.g., ‘ $M \vee \sim M$ ’) or an imperative sentence that is satisfied on every interpretation (an *imperative tautology*; e.g., ‘ $!(M \vee \sim M)$ ’). A *contradiction* is either a declarative sentence that is false on every interpretation (a *declarative contradiction*; e.g., ‘ $M \& \sim M$ ’) or an imperative sentence that is violated on every interpretation (an *imperative contradiction*; e.g., ‘ $!(M \& \sim M)$ ’). Sentences ϕ and ψ are *logically equivalent* only if either they are both declarative or they are both imperative. Declarative sentences p and q are *logically equivalent* (in other words, p is logically equivalent to q) exactly if, for any interpretation m , p and q are either both true on m or both false on m (equivalently: p is true on m exactly if q is true on m). Imperative sentences i and j are *logically equivalent* (in other words, i is logically equivalent to j) exactly if, for any interpretation m , i and j are either both satisfied on m or both violated on m or both avoided on m (equivalently: i is satisfied on m exactly if j is satisfied on m , and i is violated on m exactly if j is violated on m). Now it can be seen that, as stated in §2.1, ‘ $!M \& !P$ ’ and ‘ $!(M \& P)$ ’ are logically equivalent: for any interpretation m , by C10 ‘ $!M \& !P$ ’ is satisfied on m exactly if both ‘ $!M$ ’ and ‘ $!P$ ’ are satisfied on m , by C8 the latter holds exactly if both ‘ M ’ and ‘ P ’ are true on m , by C3 the latter holds exactly if ‘ $M \& P$ ’ is true on m , and by C8 the latter holds exactly if ‘ $!(M \& P)$ ’ is satisfied on m (and similarly for violation on m). As I prove in the Appendix, for any imperative sentence i , there are declarative sentences p and q such that i and $\lceil p \rightarrow !q \rceil$ are logically equivalent.

Recall that the second coordinate of an interpretation is a favoring relation. Formally, a favoring relation is a three-place relation on declarative sentences (i.e., a set of ordered triples of declarative sentences) that satisfies two conditions. First, the *intensionality condition*: for any declarative sentences p , q , and r , and any declarative sentences p' , q' , and r' logically equivalent to p , q , and r respectively, the ordered triple $\langle p, q, r \rangle$ is in (i.e., is a member of) the relation exactly if $\langle p', q', r' \rangle$ is. Second, the *asymmetry condition*: for any declarative sentences p , q , and r , $\langle p, q, r \rangle$ and $\langle p, r, q \rangle$ are not both in the relation. Say that p favors q over r on an interpretation exactly if $\langle p, q, r \rangle$ is in the favoring relation (which is the second coordinate) of the interpretation. So, to say that the favoring relation of any interpretation satisfies the asymmetry condition is to say that, for any declarative sentences p , q , and r , p does not favor both q over r and r over q on any interpretation. Informally, a favoring relation corresponds to *comparative reasons* (e.g., reasons for you to marry Hugh rather than Hugo), so the asymmetry condition corresponds to the claim that nothing can be a reason both for b rather than d and for d rather than b . In this paper, I do not need to engage with metaphysical debates concerning the existence and the nature of reasons for action (see, e.g., Skorupski 2010): the above formal definition of a favoring relation suffices for my purposes. The point of having a favoring relation as a coordinate of an interpretation should become clear when I define semantic validity below.⁷

⁷ There is a circularity problem related to the intensionality condition: I formulated the condition in terms of logical equivalence, which I defined in terms of (truth on) interpretations, so it is circular to define favoring relations—and thus interpretations—in terms of the condition. This problem can be circumvented by complicating the semantics as follows. Define a *subinterpretation* (of the language of SPIL) as a set of sentence letters. Say that a sentence letter is true on a *subinterpretation* exactly if it is a member of the subinterpretation. Use C2-C15 to define truth, falsity, satisfaction, violation, and avoidance on m , where m is now a *subinterpretation*. Say that p and q are logically equivalent

2.3. Semantic validity

An *argument* (of the language of SPIL) is an ordered pair whose first coordinate is a non-empty finite set of sentences (the *premises* of the argument) and whose second coordinate is a sentence (the *conclusion* of the argument). A *pure declarative* argument is an argument whose premises and conclusion are all declarative sentences, and a *pure imperative* argument is an argument whose premises and conclusion are all imperative sentences. A pure declarative argument is *semantically valid* (in SPIL)—i.e., its premises *semantically entail* its conclusion; in other words, its conclusion *semantically follows* from (the set of) its premises—exactly if its conclusion is true on every interpretation on which its premises are all true. (This is equivalent to the definition of semantic validity in classical sentential logic.) Building on previous work, I similarly say that (roughly) a pure imperative argument is semantically valid (in SPIL) when, on every interpretation, its conclusion is “supported” by everything that supports its premises. Also building on previous work, I distinguish *strong* from *weak* support—and, correspondingly, strong from weak semantic validity for pure imperative arguments—as follows:

- DEFINITION 1. For any declarative sentence p , any imperative sentence i , and any interpretation m :
- (1) p *strongly supports* i on m exactly if (a) p is true on m , (b) i is not a contradiction, and (c) for any declarative sentences q and r that are not both contradictions, if (i) i is satisfied on every interpretation on which q is true and (ii) i is violated on every interpretation on which r is true, then p favors q over r on m ;
 - (2) p *weakly supports* i on m exactly if p strongly supports on m some imperative sentence j such that (a) i is satisfied on every interpretation on which j is satisfied and (b) i is avoided on all and only those interpretations on which j is avoided.

DEFINITION 2. A pure imperative argument is (1) *strongly semantically valid* (in SPIL) exactly if, for any interpretation m , every declarative sentence that *strongly* supports on m every conjunction⁸ of all premises of the argument also *strongly* supports on m the conclusion of the argument, and is (2) *weakly semantically valid* (in SPIL) exactly if, for any interpretation m , every declarative sentence that *weakly* supports on m every conjunction of all premises of the argument also *weakly* supports on m the conclusion of the argument.

lent exactly if they are either both true or both false on every *sub*interpretation. Now the intensionality condition can be formulated without appealing to interpretations, and an interpretation can be defined without circularity as an ordered pair whose first coordinate is a subinterpretation and whose second coordinate is a three-place relation on declarative sentences that satisfies the intensionality and asymmetry conditions. Finally, define truth (satisfaction, etc.) on an interpretation as truth (satisfaction, etc.) on the subinterpretation (which is the first coordinate) of the interpretation. For the sake of simplicity, I ignore these complications in the text.

⁸ Given n (≥ 1) sentences that are either all declarative or all imperative, a *conjunction* of all of them is any sentence that can be built up from them by using each of them exactly once and by applying $n - 1$ times the following rule: if ϕ and ψ are (both declarative or both imperative) sentences, then $\lceil(\phi \ \& \ \psi)\rceil$ is a sentence. See Vranas 2011: 396-8 for an explanation of why I define semantic validity in terms of supporting *conjunctions* of all premises and not in terms of supporting *every* premise. Because sentences are *finite* strings of symbols, I do not define conjunctions of infinitely many sentences (contrast Vranas 2016: 1706 n. 1); this is why I defined an argument as having finitely many premises. If one defines similarly an *argument** as having finitely or infinitely many premises, then one can say that an argument* g with infinitely many premises is weakly semantically valid exactly if so is (according to Definition 2) some argument whose (finitely many) premises are also premises of g and whose conclusion is the same as the conclusion of g (and similarly for weak *syntactic* validity; see §2.4). Compactness holds then by definition. (A similar move does not work for *strong* semantic validity, which turns out to be non-monotonic.)

Given the intensionality condition (§2.2) and the logical equivalence of any two conjunctions of all premises of an argument, supporting (strongly or weakly, on an interpretation) *some* conjunction of all premises of an argument amounts to supporting *every* conjunction of all premises of the argument. It follows that a pure imperative argument is strongly semantically valid exactly if *some* conjunction (equivalently: *every* conjunction) of its premises semantically strongly entails its conclusion, and similarly for weak semantic entailment. It also follows from the above definitions that (1) every declarative sentence that strongly supports an imperative sentence on an interpretation also weakly supports the imperative sentence on the interpretation, (2) every pure imperative argument that is strongly semantically valid is also weakly semantically valid, and (3) for any imperative sentences i and j , the pure imperative arguments from i to j and from j to i are both weakly semantically valid exactly if they are both strongly semantically valid and also exactly if i and j are logically equivalent. ((1) follows from the second part of Definition 1; see the Appendix on how to prove (2) and (3).) Informally, the distinction between strong and weak semantic validity captures a conflict of intuitions about whether, for example, “sign the letter” entails “sign or burn the letter”: one can show that ‘! S ’ weakly but not strongly semantically entails ‘! $(S \vee B)$ ’.

Directly defending the above definitions lies beyond the scope of this paper: I have extensively defended in previous work (Vranas 2011, 2016) an account of validity on which the definitions are based,⁹ and I have compared that account to alternative accounts in the literature (e.g., Charlow 2014; Kaufmann 2012; Parsons 2013). My main goal here is to develop a *syntactic* account of validity equivalent to the semantic account provided by the above definitions. I turn next to the syntactic account.

2.4. Syntactic validity

A *derivation* (in SPIL) of an imperative sentence i from (the members of) a non-empty set Γ of imperative sentences is a finite sequence of imperative sentences (called the *lines* of the derivation) such that (1) the last line of the sequence is i and (2) each line of the sequence either is (a member or) a conjunction of members of Γ or can be obtained from previous lines by applying once a replacement rule from Table 1 (§2.4.1) or a pure imperative inference rule from Table 2 or Table 3 (§2.4.2). A pure imperative argument is *weakly syntactically valid* (in SPIL) exactly if there is a derivation of its conclusion from its premises (i.e., the conclusion can be derived, or is *derivable*, from the premises). I define *strong* syntactic validity later on (in §2.4.3).

2.4.1. Replacement rules

A *subsentence* of a given sentence is any string of consecutive symbols of the given sentence that is itself a sentence. For example, the subsentences of ‘ $M \rightarrow !\sim M$ ’ are ‘ M ’, ‘ $\sim M$ ’, ‘! $\sim M$ ’, and ‘ $M \rightarrow !\sim M$ ’. (The first three are *proper* subsentences of ‘ $M \rightarrow !\sim M$ ’: they are not identical to ‘ M

⁹ I say that the definitions are “based” on my previously defended account of validity because that account is about “arguments” whose premises and conclusions are not sentences of a formal language, but are instead what imperative and declarative sentences of natural languages typically express, namely *prescriptions* (i.e., commands, requests, instructions, suggestions, etc.) and *propositions* respectively. Deviating slightly from previous work in order to keep my definition of an interpretation (§2.2) simple, I formulated Definition 1 so that it has as consequences two claims corresponding to what in previous work I understood as assumptions about favoring, namely the claims that (1) no declarative sentence strongly supports an imperative contradiction on any interpretation (cf. Assumption 1 in Vranas 2011: 433) and (2) every declarative sentence that is true on an interpretation strongly supports on that interpretation any *semantically empty* imperative sentence (cf. Vranas 2016: 1708 n. 6), namely any imperative sentence that is avoided on every interpretation (e.g., ‘ $(M \& \sim M) \rightarrow !P$ ’).

$\rightarrow !\sim M$. There are two *occurrences* of the subsentence ‘ M ’ within ‘ $M \rightarrow !\sim M$.’) All replacement rules are based on the result that two sentences are logically equivalent if they differ only in subsentences that are logically equivalent, and this result is based on the following lemma (which I prove in the Appendix):

REPLACEMENT LEMMA. For any sentences ϕ , ψ , and ψ' such that ψ is a subsentence of ϕ and ψ is logically equivalent to ψ' , ϕ is logically equivalent to any sentence that results from replacing in ϕ at least one occurrence of ψ with ψ' .

For example, because ‘ $!\sim C$ ’ is logically equivalent to ‘ $\sim!C$ ’, ‘ $O \rightarrow !\sim C$ ’ is logically equivalent to ‘ $O \rightarrow \sim!C$ ’. In a derivation, one can obtain ‘ $O \rightarrow \sim!C$ ’ from ‘ $O \rightarrow !\sim C$ ’ (and vice versa: one can obtain ‘ $O \rightarrow !\sim C$ ’ from ‘ $O \rightarrow \sim!C$ ’) by applying once a replacement rule based on the logical equivalence between ‘ $!\sim C$ ’ and ‘ $\sim!C$ ’—or on the *general* logical equivalence (i.e., on *all* logical equivalences) between $\lceil !\sim p \rceil$ and $\lceil \sim!p \rceil$ (for all declarative sentences p).¹⁰ Table 1 lists the general logical equivalences on which the replacement rules that may be applied in derivations are based, as well as the names of the corresponding replacement rules and the abbreviations of those names. (I use the metalinguistic symbol ‘ \Leftrightarrow ’ for “is logically equivalent to”, and for simplicity I omit corner quotes in the table.) To limit the proliferation of rules, each replacement rule except for the last nine rules in the table is based on at least two general logical equivalences: at least one of them is between declarative sentences (and is familiar from classical sentential logic), and at least one of them is between imperative sentences. In the table, and in what follows, p, q, r, p' , and q' are any declarative sentences, and i, j, k , and i' are any imperative sentences.

One can see from Table 1 that many general declarative logical equivalences (i.e., logical equivalences between declarative sentences) have straightforward imperative analogs. For example, de Morgan’s laws and contraposition (i.e., transposition) hold for imperative sentences. It is sometimes claimed in the literature, however, that imperative contraposition fails because, for example, “if you killed, confess” is not equivalent to “if you don’t confess, let it not be the case that you killed”. I reply that this is not really an instance of contraposition: the contrapositive of $\lceil \phi \rightarrow \psi \rceil$ is $\lceil \sim\psi \rightarrow \sim\phi \rceil$, so the contrapositive of ‘ $K \rightarrow !C$ ’ (“if you killed, confess”) is ‘ $\sim!C \rightarrow \sim K$ ’ (“don’t confess only if you didn’t kill”; cf. Castañeda 1977: 780; Fox 2012: 892), not ‘ $\sim C \rightarrow \sim!K$ ’ (“if you don’t confess, let it not be the case that you killed”). This is an example of how symbolization in an imperative formal language clears up a not uncommon confusion in the literature.¹¹

¹⁰ In general, a replacement rule based on the logical equivalence between ψ and ψ' specifies that, given any sentence ϕ that has ψ as a subsentence, one can obtain from ϕ in a derivation any sentence that results from replacing in ϕ at least one occurrence of ψ with ψ' (and vice versa, interchanging ψ with ψ'). Even more generally, a replacement rule based on *multiple* logical equivalences (as all replacement rules in Table 1 are, since they are based on *general* logical equivalences), say on the logical equivalences between ψ and ψ' and between χ and χ' , specifies that, given any sentence ϕ that has ψ or χ (or both) as a subsentence, one can obtain from ϕ in a derivation any sentence that results from replacing in ϕ either at least one occurrence of ψ with ψ' or at least one occurrence of χ with χ' (or both, provided that the occurrences of ψ and of χ do not overlap)—and vice versa, interchanging ψ with ψ' or χ with χ' (or both). For example, by applying Negated Conditional (see Table 1) once, from ‘ $(M \& \sim P) \rightarrow \sim(O \rightarrow !C)$ ’ one can obtain ‘ $\sim(M \rightarrow P) \rightarrow (O \rightarrow \sim!C)$ ’.

¹¹ I have been myself guilty of that confusion: in light of the above considerations, I renounce my definition of a contrapositive in Vranas 2011: 404 n. 45. See Vranas 2011: 404-5 n. 46 for references to imperative contraposition in the literature.

Name of rule and abbreviation		Declarative logical equivalences	Imperative logical equivalences
Double Negation	DN	$\sim\sim p \Leftrightarrow p$	$\sim\sim i \Leftrightarrow i$
Idempotence	IP	$p \& p \Leftrightarrow p$ $p \vee p \Leftrightarrow p$	$i \& i \Leftrightarrow i$ $i \vee i \Leftrightarrow i$
Commutativity	CO	$p \& q \Leftrightarrow q \& p$ $p \vee q \Leftrightarrow q \vee p$ $p \leftrightarrow q \Leftrightarrow q \leftrightarrow p$	$i \& j \Leftrightarrow j \& i$ $i \vee j \Leftrightarrow j \vee i$ $p \leftrightarrow i \Leftrightarrow i \leftrightarrow p$
Associativity	AS	$p \& (q \& r) \Leftrightarrow (p \& q) \& r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ $p \leftrightarrow (q \leftrightarrow r) \Leftrightarrow (p \leftrightarrow q) \leftrightarrow r$	$i \& (j \& k) \Leftrightarrow (i \& j) \& k$ $i \vee (j \vee k) \Leftrightarrow (i \vee j) \vee k$ $p \leftrightarrow (q \leftrightarrow i) \Leftrightarrow (p \leftrightarrow q) \leftrightarrow i$ $p \rightarrow (i \rightarrow q) \Leftrightarrow (p \rightarrow i) \rightarrow q$
Distributivity	DI	$p \vee (q \& q') \Leftrightarrow (p \vee q) \& (p \vee q')$ $p \& (q \vee q') \Leftrightarrow (p \& q) \vee (p \& q')$ $p \rightarrow (q \& q') \Leftrightarrow (p \rightarrow q) \& (p \rightarrow q')$ $p \rightarrow (q \vee q') \Leftrightarrow (p \rightarrow q) \vee (p \rightarrow q')$ $(p \vee p') \rightarrow q \Leftrightarrow (p \rightarrow q) \& (p' \rightarrow q)$ $(p \& p') \rightarrow q \Leftrightarrow (p \rightarrow q) \vee (p' \rightarrow q)$	$p \rightarrow (i \& i') \Leftrightarrow (p \rightarrow i) \& (p \rightarrow i')$ $p \rightarrow (i \vee i') \Leftrightarrow (p \rightarrow i) \vee (p \rightarrow i')$ $(p \vee p') \rightarrow i \Leftrightarrow (p \rightarrow i) \& (p' \rightarrow i)$ $(p \vee p') \rightarrow i \Leftrightarrow (p \rightarrow i) \vee (p' \rightarrow i)$
Transposition	TR	$p \rightarrow q \Leftrightarrow \sim q \rightarrow \sim p$	$p \rightarrow i \Leftrightarrow \sim i \rightarrow \sim p$ $i \rightarrow p \Leftrightarrow \sim p \rightarrow \sim i$
Negated Conditional	NC	$\sim(p \rightarrow q) \Leftrightarrow p \& \sim q$	$\sim(p \rightarrow i) \Leftrightarrow p \rightarrow \sim i$ $\sim(i \rightarrow p) \Leftrightarrow \sim i \rightarrow p$
Material Equivalence	ME	$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \& (q \rightarrow p)$ $p \leftrightarrow q \Leftrightarrow (p \& q) \vee (\sim p \& \sim q)$	$p \leftrightarrow i \Leftrightarrow (p \rightarrow i) \& (i \rightarrow p)$ $p \leftrightarrow i \Leftrightarrow (p \rightarrow i) \vee (i \rightarrow p)$
Negated Biconditional	NB	$\sim(p \leftrightarrow q) \Leftrightarrow \sim p \leftrightarrow q$ $\sim(p \leftrightarrow q) \Leftrightarrow p \leftrightarrow \sim q$	$\sim(p \leftrightarrow i) \Leftrightarrow \sim p \leftrightarrow i$ $\sim(p \leftrightarrow i) \Leftrightarrow p \leftrightarrow \sim i$
De Morgan	DM	$\sim(p \& q) \Leftrightarrow \sim p \vee \sim q$ $\sim(p \vee q) \Leftrightarrow \sim p \& \sim q$	$\sim(i \& j) \Leftrightarrow \sim i \vee \sim j$ $\sim(i \vee j) \Leftrightarrow \sim i \& \sim j$
Exportation	EX	$p \rightarrow (q \rightarrow r) \Leftrightarrow (p \& q) \rightarrow r$	$p \rightarrow (q \rightarrow i) \Leftrightarrow (p \& q) \rightarrow i$
Absorption	AB	$p \rightarrow q \Leftrightarrow p \rightarrow (p \& q)$	$p \rightarrow !q \Leftrightarrow p \rightarrow !(p \& q)$
Contradictory Conjunct	CC	$(p \& \sim p) \& q \Leftrightarrow p \& \sim p$	$!(p \& \sim p) \& i \Leftrightarrow !(p \& \sim p)$
Contradictory Disjunct	CD	$(p \& \sim p) \vee q \Leftrightarrow q$	$!(p \& \sim p) \vee !q \Leftrightarrow !q$
Tautologous Conjunct	TC	$(p \vee \sim p) \& q \Leftrightarrow q$	$!(p \vee \sim p) \& !q \Leftrightarrow !q$
Tautologous Disjunct	TD	$(p \vee \sim p) \vee q \Leftrightarrow p \vee \sim p$	$!(p \vee \sim p) \vee i \Leftrightarrow !(p \vee \sim p)$
Tautologous Antecedent	TA	$(p \vee \sim p) \rightarrow q \Leftrightarrow q$	$(p \vee \sim p) \rightarrow i \Leftrightarrow i$
Material Implication	MI	$p \rightarrow q \Leftrightarrow \sim p \vee q$	
Unconditional Negation	UN		$\sim !p \Leftrightarrow !\sim p$
Unconditional Conjunction	UC		$!p \& !q \Leftrightarrow !(p \& q)$
Unconditional Disjunction	UD		$!p \vee !q \Leftrightarrow !(p \vee q)$
Unconditional Biconditional	UB		$p \leftrightarrow !q \Leftrightarrow !(p \leftrightarrow q)$
Mixed Conjunction	MC		$!p \& (q \rightarrow !r) \Leftrightarrow !(p \& (q \rightarrow r))$
Mixed Disjunction	MD		$!p \vee (q \rightarrow !r) \Leftrightarrow !(p \vee (q \& r))$
Imperative Conjunction	IC		$(p \rightarrow !q) \& (p' \rightarrow !q') \Leftrightarrow$ $(p \vee p') \rightarrow !((p \rightarrow q) \& (p' \rightarrow q'))$
Imperative Disjunction	ID		$(p \rightarrow !q) \vee (p' \rightarrow !q') \Leftrightarrow$ $(p \vee p') \rightarrow !((p \& q) \vee (p' \& q'))$

Table 1. Replacement rules.

Table 1 indicates that some general declarative logical equivalences have no—or no straightforward—imperative analog. For example (concerning Distributivity), imperative conjunction is not distributive over imperative disjunction, and vice versa.¹² For another example (concerning Material Implication), there is no general logical equivalence between imperative conditionals and imperative disjunctions analogous to the general declarative logical equivalence between $\lceil p \rightarrow q \rceil$ and $\lceil \sim p \vee q \rceil$. As a consequence (concerning Negated Conditional), there is no general logical equivalence between negations of imperative conditionals and imperative conjunctions analogous to the general declarative logical equivalence between $\lceil \sim(p \rightarrow q) \rceil$ and $\lceil p \& \sim q \rceil$. On the contrary, negating an imperative conditional amounts to negating its imperative antecedent or consequent; informally, “if you marry, don’t procreate” negates “if you marry, procreate”.¹³

A *replacement derivation* (in SPIL) is a derivation in which each line other than the first can be obtained from the previous line by applying once a replacement rule from Table 1. As I prove in the Appendix, the set of replacement rules in Table 1 is *complete* in the following sense: any logically equivalent sentences ϕ and ψ are *replacement interderivable* (i.e., there is a replacement derivation of ψ from ϕ and there is a replacement derivation of ϕ from ψ). Given this result, and given that any imperative tautology is logically equivalent to, for example, $\lceil!(M \vee \sim M)\rceil$, the fact that I defined derivations only from *non-empty* sets of sentences does not prevent one from proving syntactically that a sentence is an imperative tautology: there is no derivation of an imperative tautology (or of any sentence) from the empty set, but there is a replacement derivation of an imperative tautology from, for example, $\lceil!(M \vee \sim M)\rceil$.

For the completeness result, not all rules in Table 1 are needed. For example, one can show that Unconditional Conjunction, Unconditional Disjunction, Mixed Conjunction, and Mixed Disjunction are redundant given Imperative Conjunction, Imperative Disjunction, Tautologous Conjunction, Tautologous Disjunction, and Tautologous Antecedent. Nevertheless, the redundant rules are useful: they make shorter derivations available. For example, by applying Mixed Conjunction one can immediately derive $\lceil!(M \& (O \rightarrow C))\rceil$ from $\lceil!M \& (O \rightarrow !C)\rceil$, but every derivation of the former sentence from the latter one in which Mixed Conjunction may not be applied has at least six lines. Here is an example of a shortest such derivation (and also an example of a replacement derivation):

- | | |
|---|--------------------------|
| 1. $!M \& (O \rightarrow !C)$ | Premise |
| 2. $((M \vee \sim M) \rightarrow !M) \& (O \rightarrow !C)$ | 1 Tautologous Antecedent |
| 3. $((M \vee \sim M) \vee O) \rightarrow !(((M \vee \sim M) \rightarrow M) \& (O \rightarrow C))$ | 2 Imperative Conjunction |

¹² For example, $\lceil!A \& (!B \vee (C \rightarrow !D))\rceil$ is not logically equivalent to $\lceil!(A \& !B) \vee (!A \& (C \rightarrow !D))\rceil$: by applying DI, MC, MD, UC, and UD (not in that order), one can show that the former sentence is logically equivalent to $\lceil!(A \& (B \vee (C \& D)))\rceil$ but the latter sentence is logically equivalent to $\lceil!(A \& (B \vee (C \rightarrow D)))\rceil$. A special case of distributivity can be shown to hold, however: $\lceil i \vee (!q \& !q) \rceil$ is logically equivalent to $\lceil (i \vee !q) \& (i \vee !q) \rceil$, and $\lceil i \& (!q \vee !q) \rceil$ is logically equivalent to $\lceil (i \& !q) \vee (i \& !q) \rceil$.

¹³ (1) Note that $\lceil(M \rightarrow !P) \& \sim(M \rightarrow !P)\rceil$, which is logically equivalent to $\lceil M \rightarrow !(P \& \sim P)\rceil$ (as one can show by applying NC, DI, UN, and UC), is *not* an imperative contradiction, namely an imperative sentence that is *violated* on every interpretation (cf. Charlow 2014: 628), but is instead what may be called an *imperative conditional contradiction*, namely an imperative sentence that is *non-satisfied* (i.e., violated or avoided) on every interpretation. (One may similarly define an *imperative conditional tautology* as an imperative sentence that is *non-violated* on every interpretation; e.g., $\lceil M \rightarrow !(P \vee \sim P)\rceil$.) (2) For another disanalogy between declarative and imperative logical equivalences, note that the conjunctions and the disjunctions of certain imperative conditionals are logically equivalent; for example, compare the last two general imperative logical equivalences on which Distributivity is based, or those on which Material Equivalence is based. See Vranas 2008: 541-2 for discussion.

- | | |
|--|--------------------------|
| 4. $(M \vee \sim M) \rightarrow !(((M \vee \sim M) \rightarrow M) \& (O \rightarrow C))$ | 3 Tautologous Disjunct |
| 5. $!(((M \vee \sim M) \rightarrow M) \& (O \rightarrow C))$ | 4 Tautologous Antecedent |
| 6. $!(M \& (O \rightarrow C))$ | 5 Tautologous Antecedent |

2.4.2. Inference rules

Table 2 lists eight pure imperative inference rules that may be applied in derivations. All but the first two rules are straightforward analogs of familiar pure declarative inference rules,¹⁴ but the first two rules deserve elaboration. According to the first part of Strengthening the Antecedent (SA), from an imperative sentence i one can obtain in a derivation $\ulcorner p \rightarrow i \urcorner$ for any declarative sentence p . For example, from ' $O \rightarrow !C$ ' one can obtain ' $M \rightarrow (O \rightarrow !C)$ ' (which is logically equivalent to ' $(M \& O) \rightarrow !C$ '; so the antecedent of ' $O \rightarrow !C$ ', namely ' O ', is "strengthened" to ' $M \& O$ '). The first part of SA is redundant given Tautologous Antecedent, Distributivity, and Imperative Conjunction Elimination. To see this, here is a derivation of ' $M \rightarrow (O \rightarrow !C)$ ' from ' $O \rightarrow !C$ ':

- | | |
|--|--------------------------------------|
| 1. $O \rightarrow !C$ | Premise |
| 2. $(M \vee \sim M) \rightarrow (O \rightarrow !C)$ | 1 Tautologous Antecedent |
| 3. $(M \rightarrow (O \rightarrow !C)) \& (\sim M \rightarrow (O \rightarrow !C))$ | 2 Distributivity |
| 4. $M \rightarrow (O \rightarrow !C)$ | 3 Imperative Conjunction Elimination |

This kind of derivation can be used to defend the first part of SA; see Vranas 2011: 400-2 for a detailed defense. Now according to the second part of SA, if p' (semantically) entails p (§2.3), then from $\ulcorner p \rightarrow i \urcorner$ one can obtain in a derivation $\ulcorner p' \rightarrow i \urcorner$. For example, since ' $M \leftrightarrow (M \rightarrow O)$ ' entails ' O ' (as one can check by using a truth table or a sound and complete natural deduction system for classical sentential logic), from ' $O \rightarrow !C$ ' one can obtain ' $(M \leftrightarrow (M \rightarrow O)) \rightarrow !C$ '. The first part of SA is redundant given the second part and Tautologous Antecedent: from i one can obtain $\ulcorner (p \vee \sim p) \rightarrow i \urcorner$ by Tautologous Antecedent, and from this one can obtain $\ulcorner p \rightarrow i \urcorner$ by the second part of SA (because p entails $\ulcorner p \vee \sim p \urcorner$). Conversely, one can show that the second part of SA is redundant given the first part and the replacement rules in Table 1.¹⁵ So both parts of SA are redundant given Imperative Conjunction Elimination and the replacement rules. De-

¹⁴ If an inference rule has multiple parts (as SA, WC, ICE, and IDS in Table 2 do), then to apply the rule once is to apply a part of the rule once. If (a part of) an inference rule has multiple premises (as ICI, IMP, IMT, and IDS in Table 2 do), then the order of the premises does not matter for the purpose of applying the rule, and to obtain its conclusion in a derivation (by applying the rule once) its premises must be any distinct previous lines of the derivation (for example, ' $!A \& !A$ ' can be obtained by ICI from two previous lines that are both ' $!A$ ' but not from a single previous line that is ' $!A$ '). Note the absence of "Imperative Disjunctive Addition" from Table 2: from i one cannot obtain $\ulcorner i \vee j \urcorner$. For example, ' $M \rightarrow !P$ ' does not weakly semantically entail ' $(M \rightarrow !P) \vee (\sim M \rightarrow !P)$ ' (which, by Distributivity and Tautologous Antecedent, is replacement interderivable with ' $!P$ '). (Informally: "if you marry, procreate" does not entail "if you marry, procreate, or if you don't marry, procreate"—which is equivalent to "procreate".) Finally, concerning IMP and IMT, from $\ulcorner i \rightarrow p \urcorner$ and i one cannot obtain $\ulcorner !p \urcorner$, and from $\ulcorner p \rightarrow i \urcorner$ and $\ulcorner \sim i \urcorner$ one cannot obtain $\ulcorner \sim !p \urcorner$. For example (concerning IMT), ' $M \rightarrow (C \rightarrow !P)$ ' and ' $\sim(C \rightarrow !P)$ ' (which are logically equivalent to ' $(M \& C) \rightarrow !P$ ' and ' $C \rightarrow \sim !P$ ' respectively) do not weakly semantically entail ' $\sim !M$ '. (Informally: "if you marry and copulate, procreate" and "if you copulate, don't procreate" do not entail "don't marry".)

¹⁵ Indeed: from $\ulcorner p \rightarrow i \urcorner$ one can obtain $\ulcorner p' \rightarrow (p \rightarrow i) \urcorner$ by the first part of SA, and from this one can obtain $\ulcorner (p' \& p) \rightarrow i \urcorner$ by Exportation. But if p' entails p , then ' $p' \& p$ ' is logically equivalent to p , and (by the Replacement Lemma) $\ulcorner (p' \& p) \rightarrow i \urcorner$ is logically equivalent to $\ulcorner p \rightarrow i \urcorner$. Then, given the completeness of the set of replacement rules in Table 1, by applying those rules one can derive $\ulcorner p' \rightarrow i \urcorner$ from $\ulcorner (p' \& p) \rightarrow i \urcorner$.

spite being redundant, however, SA is useful because by applying it one can immediately derive, for example, $(O \& C) \rightarrow !P$ from $!P$ and from $(O \vee C) \rightarrow !P$.

According to the first part of Weakening the Consequent (WC), if q (semantically) entails q' , then from $\lceil !q \rceil$ one can obtain in a derivation $\lceil !q' \rceil$. For example:

- | | |
|---|----------------------------|
| 1. $!(M \leftrightarrow (M \rightarrow O))$ | Premise |
| 2. $!O$ | 1 Weakening the Consequent |

Clearly, to apply WC in the above example one must first check that $'M \leftrightarrow (M \rightarrow O)'$ entails $'O'$. Instead of checking this separately, it might be more perspicuous to check it by including in the derivation extra lines, namely *declarative* sentences corresponding to deriving $'O'$ from $'M \leftrightarrow (M \rightarrow O)'$ by applying pure declarative inference (and declarative replacement) rules. (Similar remarks apply to the second parts of SA and of WC.) I have no problem with this alternative approach, but in this section I defined a derivation as a sequence of only *imperative* sentences mainly to keep things simple.

Name of rule and abbreviation		Rule
Strengthening the Antecedent	SA	$\frac{i}{p \rightarrow i} \quad \text{If } p' \text{ entails } p:$ $\frac{p \rightarrow i}{p' \rightarrow i}$
Weakening the Consequent	WC	$\frac{!q}{!q'} \quad \text{If } q \text{ entails } q':$ $\frac{p \rightarrow !q}{p \rightarrow !q'}$
Imperative Conjunction Introduction	ICI	$\frac{i}{j} \quad \frac{j}{i \& j}$
Imperative Conjunction Elimination	ICE	$\frac{i \& j}{i} \quad \frac{i \& j}{j}$
Imperative Modus Ponens	IMP	$\frac{p \rightarrow i}{!p} \quad \frac{!p}{i}$
Imperative Modus Tollens	IMT	$\frac{i \rightarrow p}{\sim !p} \quad \frac{\sim !p}{\sim i}$
Imperative Disjunctive Syllogism	IDS	$\frac{i \vee j}{\sim i} \quad \frac{i \vee j}{\sim j}$ $\frac{\sim i}{j} \quad \frac{\sim j}{i}$
Ex Contradictione Quodlibet	ECQ	$\frac{!(p \& \sim p)}{i}$

Table 2. Pure imperative inference rules.

Both parts of WC are redundant given Imperative Conjunction Elimination and the replacement rules in Table 1. Here is why, concerning the second part of WC. If q entails q' , then $\lceil q' \& q \rceil$ is logically equivalent to q , and then (by the Replacement Lemma) $\lceil p \rightarrow !(q' \& q) \rceil$ is logically equivalent to $\lceil p \rightarrow !q \rceil$. Then, given the completeness of the set of replacement rules in Table 1, by applying those rules one can derive $\lceil p \rightarrow !(q' \& q) \rceil$ from $\lceil p \rightarrow !q \rceil$. Finally, from $\lceil p \rightarrow !(q'$

$\& q$)¹ one can derive $\lceil(p \rightarrow !q) \& (p \rightarrow !q)\rceil$ by Unconditional Conjunction and Distributivity, and from this one can obtain $\lceil p \rightarrow !q \rceil$ by Imperative Conjunction Elimination. Similar remarks apply to the first part of WC (which, as one can show, is also redundant given the second part of WC and Tautologous Antecedent). Despite being redundant, however, WC (just like SA) is useful because it makes shorter derivations available.

Pure declarative inference rules			Pure imperative inference rules			
Name/abbreviation	Rule		Name/abbreviation	Rule		
Conjunction Introduction	CI	$\frac{p}{p \& q}$ $\frac{q}{p \& q}$	Unconditional Conjunction Introduction	UCI	$\frac{!p}{!(p \& q)}$ $\frac{!q}{!(p \& q)}$ $\frac{!p \quad !q}{!(p \& q)}$	
Conjunction Elimination	CE	$\frac{p \& q}{p}$ $\frac{p \& q}{q}$	Unconditional Conjunction Elimination	UCE	$\frac{!(p \& q)}{!p}$ $\frac{!(p \& q)}{!q}$ $\frac{!(p \& q) \quad !p \& !q}{!q}$ $\frac{!(p \& q) \quad !p \& !q}{!p}$	
Modus Ponens	MP	$\frac{p \rightarrow q \quad p}{q}$	Unconditional Modus Ponens	UMP	$\frac{!(p \rightarrow q) \quad !p}{!q}$ $\frac{!p \rightarrow q \quad !p}{!q}$ $\frac{p \rightarrow !q \quad !p}{!q}$	
Modus Tollens	MT	$\frac{p \rightarrow q \quad \sim q}{\sim p}$	Unconditional Modus Tollens	UMT	$\frac{!(p \rightarrow q) \quad !\sim q}{!\sim p}$ $\frac{!p \rightarrow q \quad !\sim q}{!\sim p}$ $\frac{p \rightarrow !q \quad !\sim q}{!\sim p}$ $\frac{!(p \rightarrow q) \quad !\sim q}{!\sim p}$ $\frac{!(p \rightarrow q) \quad !\sim q}{!\sim p}$ etc.	
Disjunctive Syllogism	DS	$\frac{p \vee q \quad \sim p}{q}$ $\frac{p \vee q \quad \sim q}{p}$	Unconditional Disjunctive Syllogism	UDS	$\frac{!(p \vee q) \quad !\sim p}{!q}$ $\frac{!p \vee !q \quad !\sim p}{!q}$ $\frac{!(p \vee q) \quad !\sim p}{!q}$ $\frac{!p \vee !q \quad !\sim p}{!q}$ etc.	
Disjunctive Addition	DA	$\frac{p}{p \vee q}$ $\frac{p}{q \vee p}$	Unconditional Disjunctive Addition	UDA	$\frac{!p}{!(p \vee q)}$ $\frac{!p}{!p \vee !q}$ $\frac{!p}{!(q \vee p)}$ $\frac{!p}{!q \vee !p}$	
Disjunctive Dilemma	DD	$\frac{p \rightarrow q \quad r \rightarrow t \quad p \vee r}{q \vee t}$	Unconditional Disjunctive Dilemma	UDD	$\frac{!(p \rightarrow q) \quad !(r \rightarrow t) \quad !(p \vee r)}{!(q \vee t)}$ $\frac{!p \rightarrow q \quad !r \rightarrow t \quad !p \vee r}{!q \vee t}$ $\frac{p \rightarrow !q \quad r \rightarrow t \quad p \vee r}{!q \vee t}$ $\frac{!(p \rightarrow q) \quad !(r \rightarrow t) \quad !(p \vee r)}{!(q \vee t)}$ $\frac{!(p \rightarrow q) \quad !(r \rightarrow t) \quad !(p \vee r)}{!(q \vee t)}$ etc.	

Table 3. Pure declarative inference rules and their unconditional imperative counterparts.

As I prove in the Appendix, the set consisting of the replacement rules in Table 1 and of the inference rules in Table 2 is *complete*: any imperative sentence that weakly semantically follows from given imperative sentences can be derived from those sentences by applying only replacement rules from Table 1 and only inference rules from Table 2. (This differs from the sense in which, as I said in §2.4.1, the set of replacement rules in Table 1 is complete: that sense of completeness is about logically equivalent sentences, but the present sense of completeness is about weakly semantically valid pure imperative arguments.) For the purpose of making shorter derivations available, however, the pure imperative inference rules in Table 3 are useful (despite being redundant).¹⁶ The parts of those pure imperative inference rules are what I call *unconditional imperative counterparts* of the pure declarative inference rules in Table 3. Take any (or any part)

¹⁶ Some parts of the pure imperative inference rules in Table 3 do not make shorter derivations available. For example, to immediately derive $\lceil !p \& !q \rceil$ from $\lceil !p \rceil$ and $\lceil !q \rceil$ one does not need to apply the second part of UCI: one can apply ICI instead. By contrast, other parts of the pure imperative inference rules in Table 3 do make shorter derivations available. For example, $\lceil !(p \& q) \rceil$ can be derived from $\lceil !p \rceil$ and $\lceil !q \rceil$ either immediately, by the first part of UCI, or in two steps, by applying first ICI (to obtain $\lceil !p \& !q \rceil$) and then UC.

of those pure declarative inference rules and, treating p , q , r , and t as if they had no proper subsentences, prefix any combination of subsentences of each premise and of the conclusion of the rule with ‘!’ so that the resulting premises and conclusion are all imperative sentences; what results is an unconditional imperative counterpart of the pure declarative inference rule. For example, take Modus Ponens. There are three ways to prefix subsentences of $\lceil p \rightarrow q \rceil$ (i.e., the first premise) with ‘!’ so as to get an imperative sentence, resulting in the three imperative sentences $\lceil!(p \rightarrow q)\rceil$, $\lceil!p \rightarrow q\rceil$, and $\lceil p \rightarrow !q\rceil$. There is only one way to prefix subsentences of p (i.e., the second premise) with ‘!’ so as to get an imperative sentence, resulting in $\lceil!p\rceil$. Finally, there is only one way to prefix subsentences of q (i.e., the conclusion) with ‘!’ so as to get an imperative sentence, resulting in $\lceil!q\rceil$. So there are three unconditional imperative counterparts of Modus Ponens, listed in Table 3, and together they make up Unconditional Modus Ponens. Similarly for the remaining rules in Table 3. Due to limitations of space, for certain pure imperative inference rules in Table 3 only some of the parts are listed: there are eight unlisted parts of UMT, four of UDS, and 32 of UDD.¹⁷

2.4.3. Strong syntactic validity

A *strong derivation* (in SPIL) of an imperative sentence i from (the members of) a non-empty finite set Γ of imperative sentences is a finite sequence of imperative sentences such that (1) the last line of the sequence is i and (2) each line of the sequence either is a conjunction of all members of Γ or can be obtained from a previous line by applying once Strengthening the Antecedent, Ex Contradictione Quodlibet, or a replacement rule from Table 1. A pure imperative argument is *strongly syntactically valid* (in SPIL) exactly if there is a strong derivation of its conclusion from its premises. Every strong derivation is a derivation, so every *strongly* syntactically valid pure imperative argument is also *weakly* syntactically valid. Moreover, every replacement derivation is a strong derivation.

Note two differences between derivations and strong derivations. First, all pure imperative inference rules in Table 2 and in Table 3 may be applied in a derivation, but the only inference rules that may be applied in a strong derivation are Strengthening the Antecedent and Ex Contradictione Quodlibet: one can show that each of the remaining pure imperative inference rules has instances that are not strongly semantically valid arguments (except for ICI and UCI, but it turns out that these two rules would be useless in strong derivations). Second, any single premise can be the first line of a derivation, but no single premise (as opposed to a conjunction of all premises) can be the first line of a strong derivation (unless there is only one premise). This is because, as one can show, a set of imperative sentences does not always strongly—but does always weak-

¹⁷ Here is why this method of getting unconditional imperative counterparts of pure declarative inference rules works. For any pure declarative inference rule in Table 3, if q is a conjunction of all premises of (an instance of) the rule and q' is the conclusion of the rule, then q entails q' , so (1) $\lceil!q'\rceil$ is derivable from $\lceil!q\rceil$ (by WC); moreover, for any unconditional imperative counterpart of the rule, (2) its conclusion is replacement interderivable with $\lceil!q'\rceil$ (by UN or UD), and (3) a conjunction of its premises is replacement interderivable with $\lceil!q\rceil$ (for example, concerning the third part of UMT, $\lceil!\sim q \ \& \ (p \rightarrow !q)\rceil$ is replacement interderivable with $\lceil!(\sim q \ \& \ (p \rightarrow q))\rceil$ by MC). Note that (3) relies on the fact that, for any pure declarative inference rule in Table 3, at least one premise of any unconditional imperative counterpart of the rule is replacement interderivable with an unconditionally prescriptive sentence—hence the label “unconditional”. This fails for example for Hypothetical Syllogism, a pure declarative inference rule not in Table 3: it turns out that five (out of 27) “unconditional imperative counterparts” of Hypothetical Syllogism have instances that are *not* weakly semantically valid arguments. For example, $\lceil p \rightarrow !q\rceil$ and $\lceil q \rightarrow !r\rceil$ (which are *not* unconditionally prescriptive sentences) do *not* weakly semantically entail $\lceil!(p \rightarrow r)\rceil$.

ly—semantically entail every sentence in the set; for example, $\{!M, !P\}$ does not strongly—but does weakly—semantically entail ‘ $!M$ ’ (see Vranas 2011: 397).

Here is an example of a strong derivation (of ‘ $M \rightarrow !P$ ’ from ‘ $!(M \rightarrow P)$ ’):

1. $!(M \rightarrow P)$	Conjunction of all premises
2. $M \rightarrow !(M \rightarrow P)$	1 Strengthening the Antecedent
3. $M \rightarrow !(M \& (M \rightarrow P))$	2 Absorption
4. $M \rightarrow !(M \& (\sim M \vee P))$	3 Material Implication
5. $M \rightarrow !((M \& \sim M) \vee (M \& P))$	4 Distributivity
6. $M \rightarrow !(M \& P)$	5 Contradictory Disjunct
7. $M \rightarrow !P$	6 Absorption

2.5. Soundness and completeness

As I prove in the Appendix, a pure imperative argument is (1) strongly *semantically* valid exactly if it is strongly *syntactically* valid, and is (2) weakly *semantically* valid exactly if it is weakly *syntactically* valid.¹⁸ This result provides an indirect defense of my definitions of strong and weak semantic validity in §2.3, assuming that the replacement and inference rules I used to define strong and weak syntactic validity capture all “natural” patterns of sentential pure imperative reasoning.¹⁹

3. Sentential modal imperative logic (SMIL)

Does “it is impossible for you to win” entail “don’t try to win”? Here are two reasons why this question lies beyond the scope of the logic examined in the previous section, namely SPIL. First,

¹⁸ Hansen (2014) provides an alternative sound and complete natural deduction system for SPIL. More precisely, Hansen considers a language of SPIL in which every imperative sentence is either of the form $\lceil !q \rceil$ or of the form $\lceil p \rightarrow !q \rceil$ (Hansen uses ‘ \Rightarrow ’ instead of ‘ \rightarrow ’). This limitation is not crucial: as I prove in the Appendix, every imperative sentence of the language of SPIL is inderivable with a sentence of the form $\lceil p \rightarrow !q \rceil$ by using only replacement rules (which Hansen does not introduce, although in effect he relies on TA and one of his inference rules corresponds to IC). Hansen’s system has six inference rules; five of them correspond to (special cases of) SA, WC, ECQ, and IC, but the remaining rule is new. (Only the rule that corresponds to a special case of WC may not be applied in Hansen’s “strong deductions”, which roughly correspond to strong derivations.) Here is the new rule (which Hansen calls “Contextual Extensionality”) in my notation: if p entails $\lceil q \leftrightarrow r \rceil$, then from $\lceil p \rightarrow !q \rceil$ one can obtain in a (strong) derivation $\lceil p \rightarrow !r \rceil$. Although this rule has no analog in my system (but see MEC in §3.4.1), its effects can be simulated by using only replacement rules: if p entails $\lceil q \leftrightarrow r \rceil$, then $\lceil p \& q \rceil$ and $\lceil p \& r \rceil$ are logically equivalent and thus replacement interderivable, and thus so are $\lceil p \rightarrow !(p \& q) \rceil$ and $\lceil p \rightarrow !(p \& r) \rceil$, and so are also (by AB) $\lceil p \rightarrow !q \rceil$ and $\lceil p \rightarrow !r \rceil$. Although including this rule in my system would make shorter derivations available, I did not include it as an inference rule because I find it less “natural” than the rules in Table 2, and I did not include a corresponding replacement rule because it would be of a different form from the rules in Table 1.

¹⁹ One might claim that I have not really presented a “natural deduction” system, for two reasons: (1) my system does not include Conditional Proof, and (2) my system does not include two rules for each connective, an introduction rule and an elimination rule. Concerning (1), I reply that I could introduce a restricted version of Conditional Proof; see the end of note 41. Concerning (2), in a survey of natural deduction systems, Pelletier notes that “there are many systems we happily call natural deduction which do not have rules organized in this manner” (1999: 2, 2000: 106). Pelletier also argues that including only “natural” rules of inference is not sufficient for being a natural deduction system: a “system with only modus ponens as a rule of inference obeys the restriction that all the rules of inference are ‘natural’, yet no one wants to call such a system ‘natural deduction,’ so it is not a sufficient condition” (1999: 3, 2000: 107). This example, however, provides no reason to deny that I have presented a natural deduction system.

SPIL does not include modal operators. Second, SPIL does not deal with arguments from declarative premises to imperative conclusions. The logic I examine in the present section, namely SMIL, does away with these two limitations of SPIL: SMIL includes modal operators and deals not only with *pure* arguments (i.e., arguments whose premises and conclusions are either all declarative or all imperative) but also with *mixed* (i.e., non-pure) ones.

3.1. Syntax

The syntax of SMIL closely parallels the syntax of SPIL. The symbols of the language of SMIL are the symbols of the language of SPIL (§2.1) plus the modal operators ‘ \Box ’ (“it is necessary that”) and ‘ \Diamond ’ (“it is possible that”), and the formation rules of the language of SMIL for declarative and imperative sentences are the formation rules of the language of SPIL (§2.1) plus the following formation rule: if p is a declarative sentence, then $\Box p$ and $\Diamond p$ are declarative sentences. Here are two examples of how imperative English sentences can be symbolized in (the language of) SMIL:

Let it be possible for you to procreate: $\Diamond P$

Don’t copulate if it is impossible for you to ovulate: $\Box \sim O \rightarrow \sim !C$

3.2. Semantics

An *interpretation* of the language of SMIL is an ordered quadruple whose four coordinates are: first, a non-empty set whose members are called the *worlds* of the interpretation; second, a two-place relation on—i.e., a set of ordered pairs of—worlds of the interpretation (called the *accessibility relation* of the interpretation); third, a function that assigns to every world of the interpretation a set of sentence letters (namely—see below—those sentence letters that are true at the world on the interpretation); and fourth, a function that assigns to every world of the interpretation a favoring relation (defined as in §2.2, namely as a three-place relation on declarative sentences that satisfies the intensionality and asymmetry conditions). Declarative sentences are true or false *at worlds* on interpretations, and imperative sentences are satisfied, violated, or avoided *at worlds* on interpretations. Specifically, replace C1-C15 (§2.2) with C1*-C15* below, and add C16* and C17* below to deal with the modal operators:

(C1*) A sentence letter is true at a world w on an interpretation m iff it is a member of the set of sentence letters that the third coordinate of m assigns to w .

(C2*) $\neg p$ is true at w on m iff p is not true at w on m .

...

(C15*) i is avoided at w on m iff i is neither satisfied nor violated at w on m .

(C16*) $\Box p$ is true at w on m iff p is true at w' on m for every w' accessible from w on m (i.e., every w' such that $\langle w, w' \rangle$ is in the accessibility relation of m).

(C17*) $\Diamond p$ is true at w on m iff p is true at w' on m for some w' accessible from w on m .

(The formulations of C2*-C15* are obtained by inserting “at w ” before every occurrence of “on m ” in the formulations of C2-C15.) Definitions of a (declarative or imperative) tautology, a (declarative or imperative) contradiction, and logical equivalence (between declarative or between imperative sentences) can be given that are straightforward variants of the definitions in §2.2. For example, an *imperative contradiction* is an imperative sentence i such that, for any interpretation m and any world w of m , i is violated at w on m . Similarly, imperative sentences i and j are *logically equivalent* exactly if, for any interpretation m and any world w of m , i and j are either both satisfied at w on m or both violated at w on m or both avoided at w on m .

3.3. Semantic validity

A *satisfaction sentence* of an imperative sentence i is any declarative sentence s such that, for any interpretation m and any world w of m , s is true at w on m exactly if i is satisfied at w on m . A *violation sentence* and an *avoidance sentence* of an imperative sentence are defined similarly. For example, if i is ' $M \rightarrow !P$ ', then ' $M \& P$ ' is a satisfaction sentence of i , ' $M \& \sim P$ ' is a violation sentence of i , and ' $\sim M$ ' is an avoidance sentence of i . (Every imperative sentence has a satisfaction, a violation, and an avoidance sentence: as I prove in the Appendix, for any imperative sentence i , there are declarative sentences p and q such that i and ' $\Box p \rightarrow !q$ ' are logically equivalent, and then ' $\Box p \& q$ ' is a satisfaction sentence of i , ' $\Box p \& \sim q$ ' is a violation sentence of i , and ' $\sim \Box p$ ' is an avoidance sentence of i . Any two satisfaction sentences of i are logically equivalent, and similarly for violation and avoidance.) Given these definitions, I define strong and weak support at a world on an interpretation as follows:

DEFINITION 3. For any declarative sentence p , any imperative sentence i , any interpretation m , and any world w of m :

(1) p *strongly supports* i at w on m exactly if (a) p is true at w on m , (b) for some violation sentence v of i , ' $\Box v$ ' is not true at w on m , and (c) for any declarative sentences q and r such that ' $\Box \sim q$ ' and ' $\Box \sim r$ ' are not both true at w on m , if (i) ' $\Box(q \rightarrow s)$ ' is true at w on m for some satisfaction sentence s of i and (ii) ' $\Box(r \rightarrow v)$ ' is true at w on m for some violation sentence v of i , then p favors q over r at w on m (i.e., $\langle p, q, r \rangle$ is in the favoring relation that the fourth coordinate of m assigns to w);

(2) p *weakly supports* i at w on m exactly if p strongly supports at w on m some imperative sentence i' such that (a) ' $\Box(s' \rightarrow s)$ ' is true at w on m for some satisfaction sentences s and s' of i and i' respectively and (b) ' $\Box(a' \leftrightarrow a)$ ' is true at w on m for some avoidance sentences a and a' of i and i' respectively.

Definition 3 closely parallels Definition 1 (§2.3). However, instead of the interpretation-*invariant* notions used in Definition 1 (e.g., the notion of q being a contradiction), corresponding interpretation-*relative* notions are used in Definition 3 (e.g., the notion of ' $\Box \sim q$ ' being true at w on m). This is in order to get the result that certain “natural” inference rules are sound. From the second part of Definition 3 it follows that every declarative sentence that strongly supports i at w on m also weakly supports i at w on m .

Say that p *guarantees* q at w on m exactly if both p and ' $\Box(p \rightarrow q)$ ' are true at w on m . I have argued in previous work (Vranas 2016: 1708-10) that there is an analogy between guaranteeing and supporting. To capture this analogy, and also to have a uniform terminology for declarative and imperative sentences, say that p *strongly sustains* ϕ at w on m exactly if p either guarantees or strongly supports ϕ at w on m (depending on whether ϕ is a declarative or an imperative sentence), and define weak sustaining similarly (by replacing ‘*strongly*’ with ‘*weakly*’). Given this terminology, and building on previous work, I define argument validity as follows:

DEFINITION 4. (1) An argument is *strongly semantically valid* (in SMIL) exactly if, for any interpretation m and any world w of m , every declarative sentence that *strongly* sustains at w on m both every conjunction of all declarative premises and every conjunction of all imperative premises of the argument also strongly sustains at w on m the conclusion of the argument. (2) Weak semantic validity is defined similarly by replacing ‘*strongly*’ with ‘*weakly*’ throughout (1).

This definition corresponds to the weakest normal system (i.e., system K) of propositional modal logic. If one modifies the definition so as to quantify only over those interpretations whose accessibility relation satisfies a constraint c (call them c -interpretations), then one obtains a definition—call it *Definition 4c*—of what may be called strong and weak semantic c -validity. For example, by quantifying only over those interpretations whose accessibility relation satisfies the constraint of being reflexive (ρ), symmetric (σ), and transitive (τ), one obtains a definition of strong and weak semantic $\rho\sigma\tau$ -validity (corresponding to system $K\rho\sigma\tau$ of propositional modal logic, better known as system S5). (This notation is adapted from Priest 2008: 36-8.) One can define similarly a (declarative or imperative) c -contradiction, as well as *logical c-equivalence* between declarative or between imperative (but not between declarative and imperative) sentences.

Definition 4 is a *general* definition of argument validity: it applies to arguments with declarative or imperative conclusions, and with only declarative, only imperative, or both declarative and imperative premises. As I prove in the Appendix, Definition 4 yields as special cases accounts of semantic validity for pure declarative and pure imperative arguments without modal operators equivalent to the accounts in §2.3. As I also prove in the Appendix, it follows from the above definitions that (1) an *imperative* argument (namely an argument whose conclusion is imperative), whether pure or mixed, is strongly semantically c -valid *only if* it is also weakly semantically c -valid (for any constraint c that entails reflexivity),²⁰ and (2) a *declarative* argument (namely an argument whose conclusion is declarative), whether pure or mixed, is strongly semantically $\rho\sigma\tau$ -valid *exactly if* it is weakly semantically $\rho\sigma\tau$ -valid. (For *pure* declarative arguments, the equivalence between strong and weak semantic c -validity for *any* c is immediate—and thus one can just talk of semantic c -validity—because for declarative sentences both strong and weak sustaining amount to guaranteeing.) Finally, in the Appendix I prove that for any *mixed* argument there is a *pure declarative* argument which is strongly semantically $\rho\sigma\tau$ -valid exactly if so is the mixed argument (and similarly for weak semantic $\rho\sigma\tau$ -validity).

I proceed now to develop a syntactic account of c -validity equivalent to the semantic account provided by the above definitions.

3.4. Syntactic validity

Take any (declarative or imperative) sentence ϕ and any non-empty set Γ of (only declarative, or only imperative, or both declarative and imperative) sentences. Given a constraint c on the accessibility relations of interpretations, a c -derivation (in SMIL) of ϕ from (the members of) Γ is a finite sequence of sentences (called the *lines* of the derivation) such that (1) the last line of the sequence is ϕ and (2) each line of the sequence either is (a member or) a conjunction of declarative or of imperative members of Γ or can be obtained from previous lines either (a) by applying once a replacement or an inference rule from SPIL (§2.4) or an applicable inference rule from

²⁰ Moreover, an imperative argument with only declarative premises is strongly semantically c -valid *exactly if* it is weakly semantically c -valid (if c entails ρ). Some of these results apparently conflict with my previous work; for example, apparently I have argued that some strongly semantically c -valid imperative arguments are not weakly semantically c -valid (Vranas 2016: 1736 n. 48). To resolve this apparent conflict, recall that my previous work is about “arguments” whose premises and conclusions are not sentences of a formal language, but are instead prescriptions and propositions (see note 9). Some of those “arguments” cannot be adequately symbolized in (the language of) SMIL, and Definition 4 returns an incorrect verdict for some of them. (Compare: the “argument” from the proposition that Boston is a city to the proposition that something is a city cannot be adequately symbolized in any language of classical sentential—as opposed to predicate—logic.) See §6 for further discussion.

Table 4 (see §3.4.1), or (b) by using natural deduction for system Kc of propositional modal logic. (Specifying rules of natural deduction for systems of propositional modal logic would constitute a digression: my focus in this paper is on pure imperative inference rules and mixed inference rules, not on pure declarative inference rules. SA and WC must be replaced with c -SA and c -WC respectively, obtained by replacing “entails” with “ c -entails” in Table 2.) Adding constraints preserves derivability in propositional modal logic, so for example every ρ -derivation (in SMIL) is also a $\rho\tau$ -derivation. An argument is *weakly syntactically c -valid* (in SMIL) exactly if there is a c -derivation of its conclusion from its premises. I define *strong syntactic c -validity* later on (in §3.4.2).

3.4.1. Mixed inference rules

Table 4 lists the mixed inference rules that may be applied in all and only those c -derivations for which c entails the constraint (on the accessibility relations of interpretations) listed in the last column of the table. For example, Mixed Modus Ponens may be applied in ρ -derivations, $\rho\tau$ -derivations, and so on (but not in σ -derivations, $\sigma\tau$ -derivations, and so on).²¹ The first four rules in the table have only *mixed-premise* arguments as instances (namely arguments with both a declarative and an imperative premise), and the last three rules in the table have only *cross-species* arguments as instances (namely arguments with only declarative premises and an imperative conclusion or with only imperative premises and a declarative conclusion). Given Modally Strengthening the Antecedent (MSA), the second part of c -SA is redundant in c -derivations: if p' c -entails p , then $\lceil \Box(p' \rightarrow p) \rceil$ can be obtained (by convention, say that it can be obtained *from any previous line*) by using natural deduction for system Kc of propositional modal logic. Similarly, c -WC is redundant given Modally Weakening the Consequent (MWC), and so is Modally Equivalent Consequent (which is included in the table because it is needed for *strong syntactic validity*). Moreover, Mixed Modus Ponens is redundant given MSA and Tautologous Antecedent. To see this, consider the following ρ -derivation:

1. $\sim M \rightarrow (!C \rightarrow O)$	Premise
2. $\Box \sim M$	Premise
3. $\Box((M \vee \sim M) \rightarrow \sim M)$	2 Tautologous Antecedent
4. $(M \vee \sim M) \rightarrow (!C \rightarrow O)$	1, 3 Modally Strengthening the Antecedent
5. $!C \rightarrow O$	4 Tautologous Antecedent

Modally Empty Imperative can be motivated by noting that, given MSA and MWC, $\lceil p \rightarrow !q \rceil$ is ρ -derivable from $\lceil \Box \sim p \rceil$ and *any* imperative sentence. To see this, consider the following ρ -derivation:

1. $M \rightarrow !O$	Premise
2. $\Box \sim P$	Premise
3. $\Box(P \rightarrow M)$	2 Propositional Modal Logic (system $K\rho$)
4. $P \rightarrow !O$	1, 3 Modally Strengthening the Antecedent
5. $P \rightarrow !(P \& O)$	4 Absorption

²¹ Whether one constraint entails another need not be obvious. For example, it turns out that $\sigma\tau$ entails ε (so PNV may be applied in $\sigma\tau$ -derivations): if the accessibility relation of an interpretation m is both symmetric and transitive, then it is also euclidean. (By definition, it is *euclidean* exactly if, for any worlds w , w' , and w'' of m , if both w' and w'' are accessible from w on m , then w'' is accessible from w' on m .) It also turns out that $\rho\varepsilon$ and $\rho\sigma\tau$ entail each other (so all and only $\rho\varepsilon$ -derivations are $\rho\sigma\tau$ -derivations).

6. $\Box((P \& O) \rightarrow C)$
7. $P \rightarrow !C$

- 2 Propositional Modal Logic (system K ρ)
- 5, 6 Modally Weakening the Consequent

Mixed Ex Contradictione Quodlibet is a straightforward cross-species analog of ECQ. Finally, concerning Possible Non-Violation (PNV), note that $\lceil p \rightarrow q \rceil$ is a *non-violation sentence* of $\lceil p \rightarrow !q \rceil$ (because $\lceil \sim(p \rightarrow q) \rceil$ is logically equivalent to $\lceil p \& \sim q \rceil$, which is a violation sentence of $\lceil p \rightarrow !q \rceil$). The following $\rho\sigma\tau$ -derivation illustrates an application of PNV:

- | | |
|--|--|
| <ol style="list-style-type: none"> 1. $!C$ 2. $\Box \Diamond C \rightarrow \Box P$ 3. $\Diamond C$ 4. $\Box P$ 5. $P \rightarrow !C$ 6. $P \rightarrow !(P \& C)$ 7. $!(P \& C)$ | <p>Premise
Premise
1 Possible Non-Violation
2, 3 Propositional Modal Logic (system K$\rho\sigma\tau$)
1 $\rho\sigma\tau$-Strengthening the Antecedent
5 Absorption
4, 6 Mixed Modus Ponens</p> |
|--|--|

Name of rule and abbreviation		Rule	Constraint
Modally Strengthening the Antecedent	MSA	$\frac{\Box(p' \rightarrow p) \quad p \rightarrow i}{p' \rightarrow i}$	Reflexivity (ρ)
Modally Weakening the Consequent	MWC	$\frac{\Box(q \rightarrow q') \quad !q \quad !q'}{\Box(q \rightarrow q') \quad p \rightarrow !q \quad p \rightarrow !q'}$	Reflexivity (ρ)
Modally Equivalent Consequent	MEC	$\frac{\Box(q \leftrightarrow q') \quad !q \quad !q'}{\Box(q \leftrightarrow q') \quad p \rightarrow !q \quad p \rightarrow !q'}$	Reflexivity (ρ)
Mixed Modus Ponens	XMP	$\frac{p \rightarrow i \quad \Box p}{i}$	Reflexivity (ρ)
Modally Empty Imperative	MEI	$\frac{\Box \sim p}{p \rightarrow !q}$	Reflexivity (ρ)
Mixed Ex Contradictione Quodlibet	XECQ	$\frac{p \& \sim p \quad i}{! (p \& \sim p) \quad q}$	Only for the first part: Reflexivity (ρ)
Possible Non-Violation	PNV	$\frac{!q \quad \Diamond q}{p \rightarrow !q \quad \Diamond(p \rightarrow q)}$	Euclideaness (ε)

Table 4. Mixed inference rules.

3.4.2. Strong syntactic validity

A *strong c-derivation* (in SMIL) of a sentence ϕ from (the members of) a non-empty finite set Γ of sentences is a finite sequence of sentences such that (1) the last line of the sequence is ϕ and (2) each line of the sequence either is a conjunction of declarative or of *all* imperative members of Γ or can be obtained from previous lines either (a) by applying once *c*-Strengthening the Antecedent, Ex Contradictione Quodlibet, a replacement rule from Table 1, or an applicable inference rule from Table 4 *except* for MWC, or (b) by using natural deduction for system K c of proposi-

tional modal logic.²² An argument is *strongly syntactically c-valid* (in SMIL) exactly if there is a strong *c*-derivation of its conclusion from its premises. Every strong *c*-derivation is a *c*-derivation, so every *strongly syntactically c-valid* argument is also *weakly syntactically c-valid*.

The differences between *c*-derivations and strong *c*-derivations in SMIL are analogous to the differences between derivations and strong derivations in SPIL (§2.4.3). The only rule from Table 4 that may not be applied in strong *c*-derivations is Modally Weakening the Consequent. Here is an example of a strong ρ -derivation:

1. $O \rightarrow !(C \vee P)$	Conjunction of all imperative premises
2. $\Box(O \leftrightarrow \sim P)$	Declarative premise
3. $O \rightarrow !(O \& (C \vee P))$	1 Absorption
4. $\Box((O \& (C \vee P)) \leftrightarrow (O \& C))$	2 Propositional Modal Logic (system K ρ)
5. $O \rightarrow !(O \& C)$	3, 4 Modally Equivalent Consequent
6. $O \rightarrow !C$	5 Absorption
7. $\Box(\sim P \rightarrow O)$	2 Propositional Modal Logic (system K ρ)
8. $\sim P \rightarrow !C$	6, 7 Modally Strengthening the Antecedent

3.5. Soundness and completeness

For any constraint *c* on the accessibility relations of interpretations that entails ρ (and, for mixed declarative arguments, also entails ε), an argument is (1) strongly *semantically c-valid* exactly if it is strongly *syntactically c-valid*, and is (2) weakly *semantically c-valid* exactly if it is weakly *syntactically c-valid*. I prove this result in the Appendix.

4. Quantified pure imperative logic (QPIL)

Does “lock the door of every office on the fifth floor” entail “if your office is on the fifth floor, lock the door of your office”? This question lies beyond the scope of the logics examined in the previous two sections, namely SPIL and SMIL: those logics do not include quantifiers. The logic I examine in the present section, namely QPIL, does away with this limitation. Like SPIL, however, QPIL deals only with pure arguments.

4.1. Syntax

The symbols of the language of QPIL are the symbols of the language of SPIL (§2.1) plus (1) the quantifiers ‘ \forall ’ and ‘ \exists ’, (2) the identity symbol ‘=’, (3) the variables ‘*x*’, ‘*y*’, ‘*z*’, ‘*x*^o’, ‘*y*^o’, ‘*z*^o’, ‘*x*^o’, ..., (4) the constants ‘*b*’, ‘*d*’, ‘*e*’, ‘*b*^o’, ..., and (5) for any $n \geq 1$, the *n*-place predicates ‘*A_n*’, ‘*B_n*’, ..., ‘*Z_n*’, ‘*A_n*^o’,²³ The *terms* (of the language of QPIL) are the variables plus the constants. The *atomic formulas* are the sentence letters plus the strings of symbols $\lceil (f = f) \rceil$ and

²² A (*strong*) *derivation* (in SMIL) is defined just like a (strong) *c*-derivation, but with K in the place of K*c* and with SA (and WC) in the place of *c*-SA (and of *c*-WC). For any *c*, a (strong) derivation is a (strong) *c*-derivation. A *replacement (c-)derivation* (in SMIL) is a (*c*-)derivation in which each line other than the first can be obtained from the previous line by applying once a replacement rule from Table 1. For any *c*, a replacement derivation is a replacement *c*-derivation and vice versa.

²³ I use only some lower-case letters for variables and constants because I use almost all other lower-case letters of the Roman alphabet as metalinguistic symbols: (1) ‘*p*’, ‘*q*’, ‘*r*’, and ‘*t*’ for declarative sentences and formulas, (2) ‘*i*’, ‘*j*’, ‘*k*’, and ‘*l*’ for imperative sentences and formulas, (3) ‘*f*’, ‘*h*’, ‘*o*’, and ‘*u*’ for terms, constants, members of domains, and variables respectively (see below in the text), (4) ‘*a*’, ‘*s*’, and ‘*v*’ for avoidance, satisfaction, and violation sentences respectively, and (5) ‘*c*’, ‘*m*’, ‘*n*’, and ‘*w*’ for constraints, interpretations, natural numbers, and worlds respectively.

$\lceil \Pi f_1 \dots f_n \rceil$ for any terms f, f', f_1, \dots, f_n and any n -place predicate Π . The *declarative formulas* are all and only those finite strings of symbols that either are atomic formulas or can be built up from atomic formulas by applying at least once the following formation rule: if p and q are declarative formulas and u is a variable, then $\lceil \sim p \rceil$, $\lceil (p \ \& \ q) \rceil$, $\lceil (p \ \vee \ q) \rceil$, $\lceil (p \ \rightarrow \ q) \rceil$, $\lceil (p \ \leftrightarrow \ q) \rceil$, $\lceil \forall u p \rceil$, and $\lceil \exists u p \rceil$ are declarative formulas. The *imperative formulas* are all and only those finite strings of symbols that can be built up from declarative formulas by applying (1) the formation rules—call them R1*-R3*—that can be formulated by replacing ‘sentence’ and ‘sentences’ with ‘formula’ and ‘formulas’ respectively everywhere in the formulations of R1-R3 in §2.1 (R1* must be applied at least once) and (2) the following formation rule:

(R4*) If u is a variable and i is an imperative formula, then $\lceil \forall u i \rceil$ and $\lceil \exists u i \rceil$ are imperative formulas.

A *formula* is either a declarative formula or an imperative formula. It follows from these definitions that a formula is imperative exactly if it contains at least one occurrence of ‘!’ and is declarative exactly if it contains no occurrence of ‘!’ (so no formula is both declarative and imperative). A *subformula* of a given formula is any string of consecutive symbols of the given formula that is itself a formula. An occurrence of a variable u in a formula ϕ is (1) *bound* in ϕ exactly if it is also an occurrence of u in a subformula of ϕ that begins with $\lceil \forall u \rceil$ or with $\lceil \exists u \rceil$, and is (2) *free* in ϕ otherwise. A (*declarative, imperative, or atomic*) *sentence* is a (declarative, imperative, or atomic) formula in which no occurrence of any variable is free. For simplicity, I usually omit subscripts and outermost parentheses, and I usually write $\lceil f = f \rceil$ instead of $\lceil (f = f) \rceil$ and $\lceil f \neq f \rceil$ instead of $\lceil \sim (f = f) \rceil$. Here are some examples of how imperative English sentences can be symbolized in (the language of)QPIL (‘ Vx ’ stands for “you vaccinate x ”, ‘ Nx ’ for “ x is a neonate”, and so on; the semicolons separate sentences of QPIL that, as it turns out, are logically equivalent):

Vaccinate everyone: $\forall x !Vx$; $! \forall x Vx$

Vaccinate every neonate: $\forall x (Nx \rightarrow !Vx)$

Vaccinate only neonates: $\forall x (!Vx \rightarrow Nx)$

Vaccinate all and only neonates: $\forall x (!Vx \leftrightarrow Nx)$; $! \forall x (Vx \leftrightarrow Nx)$

Vaccinate some neonate: $\exists x (Nx \rightarrow !Vx)$

Don’t vaccinate any neonate: $\sim \exists x (Nx \rightarrow !Vx)$; $\forall x (Nx \rightarrow \sim !Vx)$

Don’t vaccinate every neonate: $\sim \forall x (Nx \rightarrow !Vx)$; $\exists x (Nx \rightarrow \sim !Vx)$

Oxygenate and vaccinate some neonate: $\exists x (Nx \rightarrow !(Ox \ \& \ Vx))$

Oxygenate some neonate that you vaccinate: $\exists x ((Nx \ \& \ Vx) \rightarrow !Ox)$

Oxygenate at least two neonates: $\exists x \exists y (((Nx \ \& \ Ny) \ \& \ x \neq y) \rightarrow !(Ox \ \& \ Oy))$

Oxygenate exactly one neonate: $\exists x ((Nx \rightarrow !Ox) \ \& \ \forall y (Ny \rightarrow (!Oy \rightarrow x = y)))$

Inoculate at most one neonate against every disease: $\forall x \forall y ((Nx \ \& \ Ny) \rightarrow (\forall z (Dz \rightarrow !(I_{2xz} \ \& \ I_{2yz})) \rightarrow x = y))$

One might wonder why “vaccinate every neonate” was symbolized as (1) ‘ $\forall x (Nx \rightarrow !Vx)$ ’ instead of (2) ‘ $! \forall x (Nx \rightarrow Vx)$ ’. It turns out (see §4.2) that (1) is logically equivalent to (3) ‘ $\exists x Nx \rightarrow ! \forall x (Nx \rightarrow Vx)$ ’, and that (2) is *not* logically equivalent to (3): if there are no neonates, then (2)—which symbolizes (4) “let it be the case that you vaccinate every neonate”—is *satisfied* but (3)—which symbolizes (5) “if there are any neonates, let it be the case that you vaccinate every neonate”—is *avoided*. (By contrast, the *declarative* sentences ‘ $\forall x (Nx \rightarrow Vx)$ ’ and ‘ $\exists x Nx \rightarrow \forall x (Nx \rightarrow Vx)$ ’ are logically equivalent.) The distinction between (4) and (5) captures a subtle ambiguity in the English sentence (6) “vaccinate every neonate” (cf. Ludwig 1997: 39), an ambi-

guity that can be revealed by asking: what if there are no neonates? In contexts in which the answer is that the command expressed by (6) is then trivially satisfied, (6) can be paraphrased as (4) and symbolized as (2); but in contexts in which the answer is that the command expressed by (6) is then avoided (i.e., neither satisfied nor violated), (6) can be paraphrased as (5) and symbolized either as (3) or, equivalently, as (1).

Similar remarks apply to “vaccinate some neonate”: this was symbolized as ‘ $\exists x(Nx \rightarrow !Vx)$ ’—which, as it turns out (see §4.2), is logically equivalent to ‘ $\exists xNx \rightarrow !\exists x(Nx \ \& \ Vx)$ ’—but can also be symbolized as ‘ $!\exists x(Nx \ \& \ Vx)$ ’. One might argue that both symbolizations are inadequate: it turns out that neither of them (weakly or strongly, semantically or syntactically) entails ‘ $\exists xNx$ ’, but “vaccinate some neonate” can be paraphrased as (7) “there are neonates; vaccinate at least one of them” and thus (informally) entails “there are neonates”. In reply, note first that the above symbolizations need not be inadequate if “vaccinate some neonate” *presupposes* but does not *entail* “there are neonates” and thus cannot be paraphrased as (7). Nevertheless, I can grant that in some contexts the above symbolizations are inadequate and “vaccinate some neonate” should be symbolized in the same way as (7). But how should (7) be symbolized?

One might claim that it should be symbolized as ‘ $\exists x(Nx \ \& \ !Vx)$ ’, which is *not* a formula of QPIL (cf. Clarke 1973: 201; Clarke & Behling 1998: 293; Gensler 1990: 192, 1996: 186, 2002: 185). So one might claim that ‘ $\exists x(Nx \ \& \ !Vx)$ ’ *should* be a formula, and one might propose modifying my definition of a formula by adopting the following additional formation rule: if p is a declarative formula and i is an imperative formula, then ‘ $\lceil p \ \& \ i \rceil$ ’ and ‘ $\lceil i \ \& \ p \rceil$ ’ are formulas. Addressing a similar point in the context of SPIL, in §2.1 I replied in effect that such a rule is unnecessary because, for example, nothing important is lost by symbolizing the two parts of “you are not going to marry, but nevertheless procreate” separately, as ‘ $\sim M$ ’ and ‘ $!P$ ’. In the context of QPIL, however, one might find such a reply unsatisfactory: one might argue that the two parts of (7) “there are neonates; vaccinate at least one of them” cannot be symbolized separately because the second part (“vaccinate at least one of them”) “is not by itself a complete imperative [sentence], since it does not contain the referent of the pronoun [‘them’]” (Castañeda 1963: 228-9). I reply that (7) can be paraphrased as “there are neonates; if there are neonates, vaccinate at least one of them”, so the second part of (7) can be symbolized separately as ‘ $\exists xNx \rightarrow !\exists x(Nx \ \& \ Vx)$ ’ (equivalently, as ‘ $\exists x(Nx \rightarrow !Vx)$ ’). Similar remarks apply to more complex cases; for example, the English sentence (8) “there is only one neonate; vaccinate it” can be paraphrased as (9) “there is only one neonate; if there is only one neonate, vaccinate it”, so the second part of (8) can be symbolized separately as (10) ‘ $\forall x((Nx \ \& \ \forall y(Ny \rightarrow x = y)) \rightarrow !Vx)$ ’ (“for any x , if x is the only neonate, vaccinate x ”).²⁴ I conclude that the proposed additional formation rule is unnecessary.²⁵

²⁴ It turns out that (10) is logically equivalent to ‘ $\exists x((Nx \ \& \ \forall y(Ny \rightarrow x = y)) \rightarrow !Vx)$ ’ (“for some x , if x is the only neonate, vaccinate x ”). To see that (8) can be paraphrased as (9), compare: the *declarative* sentence (11) ‘ $\exists x((Nx \ \& \ \forall y(Ny \rightarrow x = y)) \ \& \ Vx)$ ’, which symbolizes “there is only one neonate; you will vaccinate it”, is logically equivalent to ‘ $\exists x(Nx \ \& \ \forall y(Ny \rightarrow x = y)) \ \& \ \forall x((Nx \ \& \ \forall y(Ny \rightarrow x = y)) \rightarrow Vx)$ ’, which symbolizes “there is only one neonate; if there is only one neonate, you will vaccinate it”. One might note that (11) is also logically equivalent to ‘ $\exists x(Nx \ \& \ \forall y(Ny \rightarrow x = y)) \ \& \ \forall x(Nx \rightarrow Vx)$ ’ (“there is only one neonate; you will vaccinate every neonate”), so by analogy one might propose symbolizing the second part of (8) as ‘ $\forall x(Nx \rightarrow !Vx)$ ’ (“vaccinate every neonate”). I reply that, informally, “vaccinate every neonate” is *satisfied* in case there are exactly two neonates and you vaccinate them both, but in such a case there is not only one neonate and thus the second part of (8) is *avoided* (and so is my proposed paraphrase of that part, namely “if there is only one neonate, vaccinate it”).

²⁵ One might grant that the proposed rule is unnecessary but might argue that the rule is desirable because it makes *simpler* symbolizations available: symbolizing “there is only one neonate; vaccinate it” as ‘ $\exists x((Nx \ \& \ \forall y(Ny \rightarrow x =$

4.2. Semantics

An *interpretation* of the language of QPIL is an ordered quadruple whose four coordinates are: first, a set of sentence letters (namely those sentence letters that are true on the interpretation); second, a favoring relation (defined as in §2.2, namely as a three-place relation on declarative sentences that satisfies the intensionality and asymmetry conditions); third, a non-empty set called the *domain* of the interpretation; and fourth, a function (called the *denotation function* of the interpretation) that assigns to every constant a member of the domain (the *referent* of the constant on the interpretation) and assigns to every n -place predicate a set of ordered n -tuples of members of the domain (the *extension* of the predicate on the interpretation). Declarative sentences are true or false on interpretations, and imperative sentences are satisfied, violated, or avoided on interpretations. Specifically, add to C1-C15 (§2.2) C18 and C19 below to deal with atomic sentences that are not sentence letters, as well as C20-C23 below to deal with the quantifiers. First, here are C18 and C19:

(C18) For any constants h and h' , $\ulcorner h = h' \urcorner$ is true on m iff the referent of h on m is the same as the referent of h' on m .

(C19) For any constants h_1, \dots, h_n and any n -place predicate Π , $\ulcorner \Pi h_1 \dots h_n \urcorner$ is true on m iff the ordered n -tuple whose coordinates are the referents of h_1, \dots, h_n on m (in that order) is a member of the extension of Π on m .

To formulate C20-C23, I introduce first some notation and terminology. Take any variable u , any constant h , any member o of the domain of m , and any formula ϕ in which no occurrence of any variable different from u is free. Let $\phi[u/h]$ be the sentence that results from replacing in ϕ every

$y)) \ \& \ !/x)$ would be simpler than symbolizing it, as I propose, in terms of both a declarative and an imperative sentence. I reply that adopting the proposed rule would create considerable complications. First, would $\ulcorner (p \ \& \ i) \urcorner$ be (1) a declarative but not an imperative formula, (2) an imperative but not a declarative formula, (3) both a declarative and an imperative formula, or (4) neither a declarative nor an imperative formula? Against (1): if, for example, $\ulcorner \sim M \ \& \ !P \urcorner$ is a declarative but not an imperative formula (and sentence), then it seems unavoidable to say that $\ulcorner \sim M \ \& \ !P \urcorner$ is true on all and only those interpretations (let us talk, for simplicity, in the context of SPIL) on which $\ulcorner \sim M \urcorner$ is true, and then it seems unavoidable to say that $\ulcorner \sim M \ \& \ !P \urcorner$ is logically equivalent to $\ulcorner \sim M \urcorner$ —an absurd result. Against (2): if $\ulcorner \sim M \ \& \ !P \urcorner$ is an imperative but not a declarative formula, then it seems unavoidable to say that $\ulcorner \sim M \ \& \ !P \urcorner$ is satisfied (or violated) on all and only those interpretations on which $\ulcorner !P \urcorner$ is satisfied (or violated), and then it seems unavoidable to say that $\ulcorner \sim M \ \& \ !P \urcorner$ is logically equivalent to $\ulcorner !P \urcorner$ —an absurd result. Against (3): if $\ulcorner \sim M \ \& \ !P \urcorner$ is both a declarative and an imperative formula, then (assuming that imperative formulas are not true or false and that declarative formulas are not satisfied, violated, or avoided) $\ulcorner \sim M \ \& \ !P \urcorner$ is not true or false and is not satisfied, violated, or avoided on any interpretation, and then I do not see what kinds of semantic properties $\ulcorner \sim M \ \& \ !P \urcorner$ could have, so I do not see how it could play any non-trivial role in a definition of semantic validity. In favor of (4): one could say that $\ulcorner \sim M \ \& \ !P \urcorner$ is a *mixed* (i.e., neither a declarative nor an imperative) formula, which is (a) true on all and only those interpretations on which $\ulcorner \sim M \urcorner$ is true and is (b) satisfied (or violated) on all and only those interpretations on which $\ulcorner !P \urcorner$ is satisfied (or violated). To say that logical equivalence applies to mixed sentences without saying that $\ulcorner \sim M \ \& \ !P \urcorner$ is logically equivalent to $\ulcorner \sim M \urcorner$ or to $\ulcorner !P \urcorner$, one could modify the definition of logical equivalence in §2.2 as follows: sentences ϕ and ψ are logically equivalent only if either they are both declarative or they are both imperative *or they are both mixed*, and *mixed* sentences ϕ and ψ are logically equivalent exactly if, for any interpretation m , ϕ is true on m exactly if ψ is true on m , ϕ is satisfied on m exactly if ψ is satisfied on m , and ϕ is violated on m exactly if ψ is violated on m . One could further say that, if ϕ is any formula and ψ is a mixed formula, then $\ulcorner (\phi \ \& \ \psi) \urcorner$ and $\ulcorner (\psi \ \& \ \phi) \urcorner$ are mixed formulas. However, one could not also recognize $\ulcorner \sim M \ \vee \ !P \urcorner$ as a mixed formula, because then it would seem unavoidable to say that $\ulcorner \sim M \ \vee \ !P \urcorner$ is logically equivalent to $\ulcorner \sim M \ \& \ !P \urcorner$ (see Starr 2013). Even more complications arise if p and i in the proposed formation rule are formulas that are not sentences, so it is unpromising to claim—as Clarke (1973: 201; Clarke & Behling 1998: 293) in effect does—that $\ulcorner \sim M \ \& \ !P \urcorner$ is not a formula but $\ulcorner \exists x(Nx \ \& \ !/x) \urcorner$ is nevertheless a formula.

occurrence of u that is free in φ with h . (If φ is a sentence, then $\varphi[u/h]$ is just φ .) Let $m[h/o]$ be the interpretation that results from replacing in m the referent of h with o . (So m and $m[h/o]$ have the same first three coordinates, and their denotation functions assign the same extensions to all predicates and the same referents to all constants different from h . If the referent of h on m is o , then $m[h/o]$ is just m .) If φ is a *declarative* formula, say that o *verifies* φ on m exactly if, for any (equivalently: for some) constant h that does not occur in φ , $\varphi[u/h]$ is true on $m[h/o]$. If φ is an *imperative* formula, say that o *satisfies* φ on m exactly if, for any (equivalently: for some) constant h that does not occur in φ , $\varphi[u/h]$ is satisfied on $m[h/o]$, and define similarly what it is for o to *violate* or to *avoid* φ on m . Letting Δ be the domain of m , here are C20-C23:

(C20) $\ulcorner \forall u p \urcorner$ is true on m iff every member of Δ verifies p on m .

(C21) $\ulcorner \exists u p \urcorner$ is true on m iff some member of Δ verifies p on m .

(C22) $\ulcorner \forall u i \urcorner$ is (a) satisfied on m iff both some member of Δ satisfies i on m and no member of Δ violates i on m , and is (b) violated on m iff some member of Δ violates i on m .

(C23) $\ulcorner \exists u i \urcorner$ is (a) satisfied on m iff some member of Δ satisfies i on m , and is (b) violated on m iff both some member of Δ violates i on m and no member of Δ satisfies i on m .

In C20-C23 (in contrast to C2-C15), p and i need not be sentences: they must be formulas such that $\ulcorner \forall u p \urcorner$, $\ulcorner \exists u p \urcorner$, $\ulcorner \forall u i \urcorner$, and $\ulcorner \exists u i \urcorner$ are sentences (i.e., in C20-C23, no occurrence of any variable different from u is free in p or in i). See Vranas 2008: 549-50 for a defense of C22 and C23 based on understanding universal and existential quantification as generalizations of conjunction and disjunction respectively. By C15, C22, and C23, $\ulcorner \forall u i \urcorner$ is avoided on m exactly if $\ulcorner \exists u i \urcorner$ is avoided on m , and also exactly if every member of Δ avoids i on m .

The definitions in §2.2 of a (declarative or imperative) tautology, a (declarative or imperative) contradiction, and logical equivalence (between declarative or between imperative sentences) carry over into QPIL. For example, imperative sentences i and j are *logically equivalent* exactly if, for any interpretation m , i and j are either both satisfied on m or both violated on m or both avoided on m .²⁶ Moreover, one can define logical equivalence between formulas that need not be sentences: if φ and ψ are (either both declarative or both imperative) formulas in which no occurrence of any variable different from u_1, \dots, u_n is free, then φ and ψ are *logically equivalent* exactly if, for any (equivalently: for some) distinct constants h_1, \dots, h_n that occur neither in φ nor in ψ , the sentences $\varphi[u_1/h_1, \dots, u_n/h_n]$ and $\psi[u_1/h_1, \dots, u_n/h_n]$ are logically equivalent (where $\varphi[u_1/h_1, \dots, u_n/h_n]$ is the sentence that results from replacing in φ every occurrence of u_1 that is free in φ with h_1 , and so on—and similarly for ψ). For example, ‘ $M \vee \sim M$ ’ is logically equivalent to ‘ $x = x$ ’ (although the former formula is a sentence but the latter one is not).

4.3. *Semantic validity*

The definitions (1) of strong and weak support on an interpretation and (2) of strong and weak semantic validity in QPIL for pure imperative arguments (call them *Definition Q1* and *Definition*

²⁶ Now it can be seen that, as stated in §4.1, ‘ $\exists x(Nx \rightarrow !Vx)$ ’ is logically equivalent to ‘ $\exists xNx \rightarrow !\exists x(Nx \ \& \ Vx)$ ’. For any interpretation m : by C12 and C8, ‘ $\exists xNx \rightarrow !\exists x(Nx \ \& \ Vx)$ ’ is satisfied on m exactly if both ‘ $\exists xNx$ ’ and ‘ $\exists x(Nx \ \& \ Vx)$ ’ are true on m ; i.e., by C21, exactly if (1) some member of the domain Δ of m verifies ‘ $Nx \ \& \ Vx$ ’ on m (and (2) some member of Δ verifies ‘ Nx ’ on m , but (2) is redundant because it follows from (1)); i.e., exactly if, for some member o of Δ and any constant h that does not occur in ‘ $Nx \ \& \ Vx$ ’, $\ulcorner Nh \ \& \ Vh \urcorner$ is true—equivalently: $\ulcorner Nh \rightarrow !Vh \urcorner$ is satisfied—on $m[h/o]$; i.e., exactly if some member of Δ satisfies ‘ $Nx \rightarrow !Vx$ ’ on m ; i.e., by C23, exactly if ‘ $\exists x(Nx \rightarrow !Vx)$ ’ is satisfied on m (and similarly for violation on m). One can similarly show that, as also stated in §4.1, ‘ $\forall x(Nx \rightarrow !Vx)$ ’ is logically equivalent to ‘ $\exists xNx \rightarrow !\forall x(Nx \rightarrow Vx)$ ’.

$Q2$ respectively) have formulations identical (except that “in SPIL” is replaced with “inQPIL”) to the formulations of the corresponding definitions in §2.3 (Definition 1 and Definition 2 respectively), although here one quantifies over sentences, interpretations, and arguments of the language ofQPIL instead of SPIL. (Similarly for the definition of semantic validity inQPIL for pure declarative arguments.) For brevity, I do not formulate Definition $Q1$ or Definition $Q2$, and I refer to the formulations in §2.3.

4.4. Syntactic validity

A *derivation* (inQPIL) of an imperative sentence i from (the members of) a non-empty set Γ of imperative sentences is a finite sequence of imperative sentences such that (1) the last line of the sequence is i and (2) each line of the sequence either is (a member or) a conjunction of members of Γ or can be obtained from previous lines by applying once a replacement rule from Table 1 (§2.4.1) or Table 5 (§4.4.1), or a pure imperative inference rule from Table 2, Table 3 (§2.4.2), or Table 6 (§4.4.2). Note that here p , i , and so on in Table 1 are understood as *formulas* (that need not be sentences). A pure imperative argument is *weakly syntactically valid* (inQPIL) exactly if there is a derivation of its conclusion from its premises. I define *strong* syntactic validity later on (in §4.4.3).

4.4.1. Replacement rules

Table 5 lists the general logical equivalences on which the new replacement rules that may be applied in derivations inQPIL are based, as well as the names of the corresponding replacement rules and the abbreviations of those names. In the table, t and t' are any terms, u and u' are any variables, p and q are any declarative formulas, i and j are any imperative formulas, $p[u/t]$ is the formula that results from replacing in p every occurrence of u that is free in p with t , $p(t/t')$ is any formula that results from replacing in p some or all occurrences of t (that are free in p , if t is a variable) with t' (provided that, if t' is a variable, all occurrences of t' in $p(t/t')$ that result from replacing occurrences of t in p are free in $p(t/t')$), and similarly for $i[u/t]$ and $i(t/t')$.

All but the first two rules in Table 5 are (straightforward extensions of rules) familiar from classical first-order logic with identity,²⁷ but the first two rules deserve elaboration. According to the general logical equivalences on which Unconditional Quantification and Imperative Quantification are based, some (imperative) formulas in which an occurrence of ‘!’ is *bound* (i.e., is also an occurrence of ‘!’ in a subformula that begins with $\lceil \forall u \rceil$ or with $\lceil \exists u \rceil$ for some variable u) are logically equivalent to formulas in which every occurrence of ‘!’ is *free* (i.e., not bound). It turns out that this holds in general: as I prove in the Appendix, for any imperative formula i , there are declarative formulas p and q such that i and $\lceil p \rightarrow !q \rceil$ are logically equivalent (and, clearly, the only occurrence of ‘!’ in $\lceil p \rightarrow !q \rceil$ is free). It follows that, if one modified my definition of a formula by dropping the formation rule R4*, there would still be enough formulas to symbolize every English sentence that can be symbolized inQPIL. A grade of “imperative involvement” analogous to the “third grade of modal involvement” (Quine 1953) is redundant inQPIL.

²⁷ Like distributivity (§2.4.1), prenex distributivity fails in general for imperative formulas. For example, $\lceil \forall x(!A \vee (Bx \rightarrow !Cx)) \rceil$ is not logically equivalent to $\lceil !A \vee \forall x(Bx \rightarrow !Cx) \rceil$: by applying MD, UQ, IQ, and PD (not in that order), one can show that the former sentence is logically equivalent to $\lceil !(A \vee \forall x(Bx \& Cx)) \rceil$ but the latter sentence is logically equivalent to $\lceil !(A \vee (\exists x Bx \& \forall x(Bx \rightarrow Cx))) \rceil$. A special case of prenex distributivity can be shown to hold, however (cf. note 12): if no occurrence of u in i is free in i , then $\lceil \forall u(i \vee !q) \rceil$ is logically equivalent to $\lceil i \vee \forall u!q \rceil$, and $\lceil \exists u(i \& !q) \rceil$ is logically equivalent to $\lceil i \& \exists u!q \rceil$.

Unconditional Quantification is redundant given Imperative Quantification, Vacuous Quantification, Tautologous Antecedent, and Tautologous Conjunct. To see this, consider the following derivation:

- | | |
|---|-----------------------------|
| 1. $\exists x!Vx$ | Premise |
| 2. $\exists x((M \vee \sim M) \rightarrow !Vx)$ | 1 Tautologous Antecedent |
| 3. $\exists x(M \vee \sim M) \rightarrow !\exists x((M \vee \sim M) \& Vx)$ | 2 Imperative Quantification |
| 4. $\exists x(M \vee \sim M) \rightarrow !\exists xVx$ | 3 Tautologous Conjunct |
| 5. $(M \vee \sim M) \rightarrow !\exists xVx$ | 4 Vacuous Quantification |
| 6. $!\exists xVx$ | 5 Tautologous Antecedent |

Name of rule and abbreviation		Declarative logical equivalences	Imperative logical equivalences
Unconditional Quantification	UQ		$\forall u!p \Leftrightarrow !\forall up$ $\exists u!p \Leftrightarrow !\exists up$
Imperative Quantification	IQ		$\forall u(p \rightarrow !q) \Leftrightarrow \exists up \rightarrow !\forall u(p \rightarrow q)$ $\exists u(p \rightarrow !q) \Leftrightarrow \exists up \rightarrow !\exists u(p \& q)$
Quantifier Negation	QN	$\sim \forall up \Leftrightarrow \exists u\sim p$ $\sim \exists up \Leftrightarrow \forall u\sim p$	$\sim \forall ui \Leftrightarrow \exists u\sim i$ $\sim \exists ui \Leftrightarrow \forall u\sim i$
Quantifier Commutativity	QC	$\forall u\forall u'p \Leftrightarrow \forall u'\forall up$ $\exists u\exists u'p \Leftrightarrow \exists u'\exists up$	$\forall u\forall u'i \Leftrightarrow \forall u'\forall ui$ $\exists u\exists u'i \Leftrightarrow \exists u'\exists ui$
Quantifier Distributivity	QD	$\forall u(p \& q) \Leftrightarrow \forall up \& \forall uq$ $\exists u(p \vee q) \Leftrightarrow \exists up \vee \exists uq$	$\forall u(i \& j) \Leftrightarrow \forall ui \& \forall uj$ $\exists u(i \vee j) \Leftrightarrow \exists ui \vee \exists uj$
Redundant Instance	RI	$\forall up \Leftrightarrow p[u/t] \& \forall up$ $\exists up \Leftrightarrow p[u/t] \vee \exists up$	$\forall ui \Leftrightarrow i[u/t] \& \forall ui$ $\exists ui \Leftrightarrow i[u/t] \vee \exists ui$
Replacing Variables	RV	If p and q are <i>similar</i> with respect to u' and u . ²⁸ $\forall u'p \Leftrightarrow \forall uq$ $\exists u'p \Leftrightarrow \exists uq$	If i and j are <i>similar</i> with respect to u' and u : $\forall u'i \Leftrightarrow \forall u'j$ $\exists u'i \Leftrightarrow \exists u'j$
Vacuous Quantification	VQ	If no occurrence of u in p is free in p : $\forall up \Leftrightarrow p$ $\exists up \Leftrightarrow p$	If no occurrence of u in i is free in i : $\forall ui \Leftrightarrow i$ $\exists ui \Leftrightarrow i$
Prenex Distributivity	PD	If no occurrence of u in p is free in p : $\forall u(p \vee q) \Leftrightarrow p \vee \forall uq$ $\exists u(p \& q) \Leftrightarrow p \& \exists uq$	
Identity Reflexivity	IR	$p \vee \sim p \Leftrightarrow t = t$	
Identity Substitution	IS	$(t = t) \& p \Leftrightarrow (t = t) \& p(t/t')$	$!(t = t) \& i \Leftrightarrow !(t = t) \& i(t/t')$

Table 5. New replacement rules.

The above is also an example of a *replacement derivation* (in QPIL), namely a derivation in which each line other than the first can be obtained from the previous line by applying once a replacement rule from Table 1 or Table 5. As I prove in the Appendix, the set of replacement rules in Table 1 and Table 5 is *complete*: any logically equivalent sentences ϕ and ψ are *replacement*

²⁸ Formulas ϕ and ψ are *similar* with respect to variables u' and u exactly if (1) ψ is the formula that results from replacing in ϕ every occurrence of u' that is free in ϕ with u , (2) no occurrence of u in ϕ is free in ϕ , and (3) no occurrence of u' in ϕ that is free in ϕ is also an occurrence of u' in a subformula of ϕ that begins with $\lceil \forall u' \rceil$ or with $\lceil \exists u' \rceil$. Less formally, (1)-(3) together amount to the claim that ϕ and ψ are the same except that ϕ has free occurrences of u' at exactly those places where ψ has free occurrences of u .

interderivable (i.e., there is a replacement derivation of ψ from ϕ and there is a replacement derivation of ϕ from ψ).

4.4.2. Inference rules

Table 6 lists the new pure imperative inference rules that may be applied in derivations in QPIL. In the Table, u is any variable, h is any constant, p and i are any (declarative and imperative, respectively) formulas in which no occurrence of any variable different from u is free, $p[u/h]$ is the sentence that results from replacing in p every occurrence of u that is free in p with h , and similarly for $i[u/h]$.

Name of rule and abbreviation		Rule
Imperative Universal Instantiation	IUI	$\frac{\forall ui}{i[u/h]}$
Imperative Existential Generalization	IEG	$\frac{!p[u/h]}{\exists u!p}$
Imperative Existential Instantiation	IEI	If h does not occur in any previous line or in the last line of the derivation: $\frac{\exists ui}{i[u/h]}$
Imperative Universal Generalization	IUG	If h does not occur in any premise or in i , and no constant that occurs in $i[u/h]$ is <i>introduced</i> by IEI (i.e., occurs first in a line that can be obtained only by applying IEI): $\frac{i[u/h]}{\forall ui}$

Table 6. New pure imperative inference rules.

Imperative Universal Instantiation (IUI) and Imperative Existential Generalization (IEG) are redundant given Strengthening the Antecedent, Weakening the Consequent, Imperative Quantification, Unconditional Quantification, and Absorption. To see that IUI is redundant, consider the following derivation:

1. $\forall x(Nx \rightarrow !Vx)$	Premise
2. $\exists xNx \rightarrow !\forall x(Nx \rightarrow Vx)$	1 Imperative Quantification
3. $Nb \rightarrow !\forall x(Nx \rightarrow Vx)$	2 Strengthening the Antecedent
4. $Nb \rightarrow !(Nb \rightarrow Vb)$	3 Weakening the Consequent
5. $Nb \rightarrow !(Nb \& (Nb \rightarrow Vb))$	4 Absorption
6. $Nb \rightarrow !(Nb \& Vb)$	5 Weakening the Consequent
7. $Nb \rightarrow !Vb$	6 Absorption

Despite being redundant, IUI and IEG are useful because they make shorter derivations available. Note that the premise of IEG is an *unconditionally* prescriptive sentence; this is because, for example, ' $Nb \rightarrow !Vb$ ' does *not* weakly semantically entail ' $\exists x(Nx \rightarrow !Vx)$ '.²⁹ Note also that one can immediately derive ' $!\exists xVx$ ' from ' $!Vb$ ' by applying Weakening the Consequent (after checking that ' Vb ' entails ' $\exists xVx$ '; in general, to perform such checks one can use for example a sound and complete natural deduction system for classical first-order logic with identity).

²⁹ One can show this by using the Equivalence Lemma for QPIL (see the Appendix). Compare the failure of "Imperative Disjunctive Addition" pointed out in note 14.

4.4.3. Strong syntactic validity

The definitions of a strong derivation in QPIL and of strong syntactic validity in QPIL have formulations identical to the formulations of the corresponding definitions in §2.4.3, except that (1) “in SPIL” is replaced with “in QPIL” and (2) here the replacement rules from Table 5 may also be applied. Every strong derivation (in QPIL) is a derivation, so every *strongly* syntactically valid pure imperative argument is also *weakly* syntactically valid.

4.5. Soundness and completeness

As I prove in the Appendix, a pure imperative argument is (1) strongly *semantically* valid exactly if it is strongly *syntactically* valid, and is (2) weakly *semantically* valid exactly if it is weakly *syntactically* valid.

5. Quantified modal imperative logic (QMIL)

Does “it is impossible for you to trust anyone who has betrayed you” entail “trust only those who have not betrayed you”? This question lies beyond the scope of the logics examined in the previous three sections: none of those logics includes *both* quantifiers and modal operators. The logic I examine in the present section, namely QMIL, does away with this limitation. Moreover, QMIL (like SMIL) deals with both pure and mixed arguments.

5.1. Syntax

The syntax of QMIL closely parallels the syntax of QPIL. The symbols of the language of QMIL are the symbols of the language of QPIL (§4.1) plus (1) the one-place predicate ‘ \mathcal{E} ’ (the *existence predicate*) and (2) the modal operators ‘ \square ’ and ‘ \diamond ’, and the formation rules of the language of QMIL for declarative and imperative formulas are the formation rules of the language of QPIL (§4.1) plus the following formation rule: if p is a declarative formula, then ‘ $\square p$ ’ and ‘ $\diamond p$ ’ are declarative formulas. Here are two examples of how imperative English sentences can be symbolized in (the language of) QMIL:

Worship only necessary beings: $\forall x (!Wx \rightarrow \square \mathcal{E}x)$

Learn everything you can possibly learn: $\forall x (\diamond Lx \rightarrow !Lx)$

5.2. Semantics

An *interpretation* of the language of QMIL is an ordered quintuple whose five coordinates are: first, a non-empty set whose members are called the *worlds* of the interpretation; second, a non-empty set called the *domain of the interpretation* (the set of all *objects*); third, a function that assigns to every world of the interpretation (a set whose members are) a set of sentence letters, a favoring relation, and a subset (called the *domain of the world*) of the domain of the interpretation; fourth, a two-place relation on worlds (the *accessibility relation* of the interpretation); and fifth, a function (the *denotation function* of the interpretation) that assigns to every constant a member of the domain of the interpretation (the *referent* of the constant on the interpretation) and assigns to every n -place predicate at every world a set of ordered n -tuples of members of the domain of the interpretation (the *extension* of the predicate at the world on the interpretation), subject to the constraint of assigning to the *existence* predicate at any world the domain of the world (so the domain of a world is the set of all objects that *exist* at the world). The domains of worlds may be empty, and the union of the domains of all worlds need not exhaust the domain of the interpretation.

Declarative sentences are true or false *at worlds* on interpretations, and imperative sentences are satisfied, violated, or avoided *at worlds* on interpretations. Specifically, add to C1*-C17* (§3.2) what I call C18*-C23*, formulated by inserting “at w ” before every occurrence of “on m ” in the formulations of C18-C23 (and of the accompanying definitions) in §4.2, except that “the referent of h on m ” is *not* replaced with “the referent of h at w on m ” (and similarly for similar expressions): unlike the extensions of predicates, the referents of constants are *not* relative to worlds. (Alternatively, one could say that a constant has the *same* referent at every world on an interpretation.) Given C20*-C23*, the quantifiers ‘ \forall ’ and ‘ \exists ’ “range” over *all* objects. One could also introduce the *restricted* quantifiers ‘ \forall^e ’ (“for any *existing* object”) and ‘ \exists^e ’ (“for some *existing* object”), and write $\lceil \forall^e u p \rceil$, $\lceil \exists^e u p \rceil$, $\lceil \forall^e u i \rceil$, and $\lceil \exists^e u i \rceil$ instead of $\lceil \forall u (\mathcal{E}u \rightarrow p) \rceil$, $\lceil \exists u (\mathcal{E}u \& p) \rceil$, $\lceil \forall u (\mathcal{E}u \rightarrow i) \rceil$, and $\lceil \exists u (\mathcal{E}u \rightarrow i) \rceil$ respectively.

5.3. Semantic validity

The definitions (1) of strong and weak support on an interpretation and (2) of strong and weak semantic validity in QMIL (call them *Definition Q3* and *Definition Q4c* respectively) have formulations identical (except that “in SMIL” is replaced with “in QMIL”) to the formulations of the corresponding definitions in §3.3 (Definition 3 and Definition 4c respectively), although here one quantifies over sentences, interpretations, and arguments of the language of QMIL instead of SMIL.

5.4. Syntactic validity

The definitions of a c -derivation, a strong c -derivation, and weak and strong syntactic c -validity in QMIL have formulations identical to the formulations of the corresponding definitions in §3.4, except that (1) “in SMIL” is replaced with “in QMIL”, (2) here the replacement rules from Table 5 and the inference rules from Table 6 may also be applied (the latter not in *strong* c -derivations), and (3) here natural deduction for systems of *quantified* (instead of *propositional*) modal logic may be used (specifically, systems that correspond to “constant domain” interpretations; see, e.g., Fitting & Mendelsohn 1998). Here is an example of a ρ -derivation (which also provides a solution to the “logic test” in §1):

1. $\forall x(!Tx \rightarrow (Mx \vee Px))$	Premise
2. $\Box \forall x \sim Px$	Premise
3. $!Tb \rightarrow (Mb \vee Pb)$	1 Imperative Universal Instantiation
4. $\sim(Mb \vee Pb) \rightarrow \sim!Tb$	3 Transposition
5. $\Box \sim Pb$	2 Quantified Modal Logic (system CK ρ ³⁰)
6. $\Box(\sim Mb \rightarrow \sim(Mb \vee Pb))$	5 Quantified Modal Logic (system CK ρ)
7. $\sim Mb \rightarrow \sim!Tb$	4, 6 Modally Strengthening the Antecedent
8. $!Tb \rightarrow Mb$	7 Transposition
9. $\forall x(!Tx \rightarrow Mx)$	8 Imperative Universal Generalization

5.5. Soundness and completeness

As I explain in the Appendix, a soundness and completeness theorem for QMIL holds that has a formulation identical to the formulation of the soundness and completeness theorem for SMIL in §3.5.

³⁰ For any c , let CK c be the system of constant domain quantified modal logic that corresponds to system K c of propositional modal logic (cf. Priest 2008: 309).

6. Conclusion

The culmination of this paper, namely Quantified Modal Imperative Logic (QMIL), has significant expressive resources: it includes both quantifiers and modal operators, and it deals with both pure and mixed arguments. Nevertheless, QMIL has also significant expressive limitations. For example, QMIL does not include *deontic* operators, and does not deal with *permissive* or *interrogative* sentences (cf. Vanderveken 1991: 10-11). Perhaps more significantly, QMIL does not have the expressive resources to adequately symbolize certain natural language sentences about *reasons*. For example, I have argued (Vranas 2016: 1716) that (the proposition expressed by) “the fact that you have sworn to tell the truth is an undefeated reason for you to tell the truth” entails (the prescription expressed by) “tell the truth”, but it turns out that no way of symbolizing these English sentences in QMIL results in an argument valid in QMIL.³¹ I think this is no more a reason to dismiss QMIL, however, than the inability of classical first-order logic to adequately symbolize “some critics admire only one another” (Boolos 1984: 432-3) is a reason to dismiss that logic. I maintain a *principle of parity*: “the standards of success for imperative logic should not be higher or lower than those for standard declarative logic” (Vranas 2011: 420). I regard the limitations of QMIL as entrances to avenues for future research.

Appendix: Theorems and proofs

A.1. Sentential pure imperative logic (SPIL)

My main goal here is to prove the following theorem:

SOUNDNESS AND COMPLETENESS THEOREM (FOR SPIL).

- (1) Concerning logical equivalence: Sentences ϕ and ψ are logically equivalent if (*soundness*) and only if (*completeness*) they are replacement interderivable.
- (2) Concerning strong and weak semantic validity: A pure imperative argument is strongly semantically valid if (*soundness*) and only if (*completeness*) it is strongly syntactically valid, and is weakly semantically valid if (*soundness*) and only if (*completeness*) it is weakly syntactically valid.

Since, as I explained in §2.4.3, every strongly *syntactically* valid pure imperative argument is also weakly syntactically valid, an immediate corollary of the Soundness and Completeness Theorem is that every strongly *semantically* valid pure imperative argument is also weakly semantically valid. Before I prove the theorem, I prove the following three lemmata:

CANONICAL FORM LEMMA (FOR SPIL). For any imperative sentence i , there are declarative sentences p and q such that i and $\lceil p \rightarrow !q \rceil$ are replacement interderivable.

REPLACEMENT LEMMA (FOR SPIL). For any sentences ϕ , ψ , and ψ' such that ψ is a subsentence of ϕ and ψ is logically equivalent to ψ' , ϕ is logically equivalent to any sentence that results from replacing in ϕ at least one occurrence of ψ with ψ' .

³¹ Indeed: for any c that entails ρ , (1) the premise of any resulting argument of QMIL is a declarative sentence t that is not a c -contradiction (because “the fact that you have sworn to tell the truth is an undefeated reason for you to tell the truth” does not express an impossible proposition), and (2) the conclusion is an imperative sentence i such that any avoidance sentence a of i is a contradiction (because “tell the truth” expresses an unconditional prescription), so (3) t does not semantically c -entail $\lceil \Box a \rceil$ (because $\lceil \Box a \rceil$ is a c -contradiction but t is not), and thus (4) the argument from t to i is neither strongly nor weakly semantically c -valid (by the Equivalence Lemma for QMIL; see the Appendix). (See also note 20.)

EQUIVALENCE LEMMA (FOR SPIL). For any imperative sentences i and j :

(1) i strongly semantically entails j exactly if either i is a contradiction or both (a) i is satisfied on every interpretation on which j is satisfied and (b) i is violated on every interpretation on which j is violated;

(2) i weakly semantically entails j exactly if both (a) j is avoided on every interpretation on which i is avoided and (b) i is violated on every interpretation on which j is violated.

A corollary of the Equivalence Lemma is that, for any imperative sentences i and j , the pure imperative arguments from i to j and from j to i are both weakly semantically valid exactly if they are both strongly semantically valid and also exactly if i and j are logically equivalent.

In §A.1.1-§A.1.3 I prove the three lemmata, and in §A.1.4 I prove the theorem.

A.1.1. Proof of the Canonical Form Lemma

The proof is by induction on the number of occurrences of connectives in an imperative sentence. For the base step, take any imperative sentence i in which no connectives occur. Then i is $\ulcorner p \urcorner$ for some sentence letter p , and then, by TA (see Table 1), i is replacement interderivable with $\ulcorner (p \vee \sim p) \urcorner \rightarrow \ulcorner p \urcorner$. For the inductive step, take any natural number n and suppose (*induction hypothesis*) that, for any imperative sentence i with at most n occurrences of connectives, there are declarative sentences p and q such that i and $\ulcorner p \rightarrow \ulcorner q \urcorner$ are replacement interderivable. To complete the proof, take any imperative sentence i with at most $n + 1$ occurrences of connectives. There are seven cases to consider.

Case 1: For some declarative sentence p , i is $\ulcorner p \urcorner$. Then, by TA, i is replacement interderivable with $\ulcorner (p \vee \sim p) \urcorner \rightarrow \ulcorner p \urcorner$.

Case 2: For some imperative sentence j , i is $\ulcorner \sim j \urcorner$. Then j has at most n occurrences of connectives and thus, by the induction hypothesis, j is replacement interderivable with $\ulcorner p \rightarrow \ulcorner q \urcorner$ (for some declarative sentences p and q ; I omit such explanations in what follows). Then i is replacement interderivable with $\ulcorner \sim(p \rightarrow \ulcorner q \urcorner) \urcorner$, and thus also, by NC and UN, with $\ulcorner p \rightarrow \ulcorner \sim q \urcorner \urcorner$.

Case 3: i is $\ulcorner j \& k \urcorner$. Then j has most n occurrences of connectives and thus, by the induction hypothesis, j is replacement interderivable with $\ulcorner p \rightarrow \ulcorner q \urcorner$. Similarly, k is replacement interderivable with $\ulcorner p' \rightarrow \ulcorner q' \urcorner$. So i is replacement interderivable with $\ulcorner (p \rightarrow \ulcorner q \urcorner) \& (p' \rightarrow \ulcorner q' \urcorner) \urcorner$, and thus also, by IC, with $\ulcorner (p \vee p') \urcorner \rightarrow \ulcorner ((p \rightarrow \ulcorner q \urcorner) \& (p' \rightarrow \ulcorner q' \urcorner)) \urcorner$.

Case 4: i is $\ulcorner j \vee k \urcorner$. Then, similarly to case 3, i is replacement interderivable with $\ulcorner (p \rightarrow \ulcorner q \urcorner) \vee (p' \rightarrow \ulcorner q' \urcorner) \urcorner$, and thus also, by ID, with $\ulcorner (p \vee p') \urcorner \rightarrow \ulcorner ((p \rightarrow \ulcorner q \urcorner) \vee (p' \rightarrow \ulcorner q' \urcorner)) \urcorner$.

Case 5: i is $\ulcorner p \rightarrow j \urcorner$. Then, by the induction hypothesis, j is replacement interderivable with $\ulcorner q \rightarrow \ulcorner r \urcorner \urcorner$. So i is replacement interderivable with $\ulcorner p \rightarrow (q \rightarrow \ulcorner r \urcorner) \urcorner$, and thus also, by EX, with $\ulcorner (p \& q) \rightarrow \ulcorner r \urcorner \urcorner$.

Case 6: i is $\ulcorner j \rightarrow p \urcorner$. Then, similarly to case 5, i is replacement interderivable with $\ulcorner (q \rightarrow \ulcorner r \urcorner) \rightarrow \ulcorner p \urcorner \urcorner$, and thus also, by TR, NC, EX, and UN, with $\ulcorner (\sim p \& q) \rightarrow \ulcorner \sim r \urcorner \urcorner$.

Case 7: i is either $\ulcorner p \leftrightarrow j \urcorner$ or $\ulcorner j \leftrightarrow p \urcorner$. Since, by CO, $\ulcorner p \leftrightarrow j \urcorner$ and $\ulcorner j \leftrightarrow p \urcorner$ are replacement interderivable, suppose i is $\ulcorner p \leftrightarrow j \urcorner$. Then, similarly to case 5, i is replacement interderivable with $\ulcorner p \leftrightarrow (q \rightarrow \ulcorner r \urcorner) \urcorner$, and thus also, by ME, with $\ulcorner (p \rightarrow (q \rightarrow \ulcorner r \urcorner)) \& ((q \rightarrow \ulcorner r \urcorner) \rightarrow p) \urcorner$. So i , by EX and case 6, is replacement interderivable with $\ulcorner ((p \& q) \rightarrow \ulcorner r \urcorner) \& ((\sim p \& q) \rightarrow \ulcorner \sim r \urcorner) \urcorner$, and thus also, by IC, with $\ulcorner ((p \& q) \vee (\sim p \& q)) \urcorner \rightarrow \ulcorner (((p \& q) \rightarrow r) \& ((\sim p \& q) \rightarrow \sim r)) \urcorner$.

A.1.2. Proof of the Replacement Lemma

I assume that the lemma holds for the case in which ϕ is a declarative sentence (since this case corresponds to a well-known result from classical logic), so suppose ϕ is an imperative sentence i . The proof is by induction on the number of occurrences of connectives in i . For the base step, take any imperative sentence i in which no connectives occur. Then i is $\ulcorner!p\urcorner$ for some sentence letter p , and if p (which is the only proper subsentence of $\ulcorner!p\urcorner$) is replaced with some sentence p' which is logically equivalent to p , then the resulting sentence, namely $\ulcorner!p'\urcorner$, is logically equivalent to $\ulcorner!p\urcorner$: for any interpretation m , $\ulcorner!p\urcorner$ is satisfied on m exactly if p is true on m (by C8; see §2.2), and thus exactly if p' is true on m (by the logical equivalence of p and p'), and thus exactly if $\ulcorner!p'\urcorner$ is satisfied on m (and similarly for violation on m). For the inductive step, take any natural number n and suppose (*induction hypothesis*) that, for any imperative sentence i with at most n occurrences of connectives, and any sentences ψ and ψ' such that ψ is a subsentence of i and ψ is logically equivalent to ψ' , i is logically equivalent to any sentence that results from replacing in i at least one occurrence of ψ with ψ' . To complete the proof, take any imperative sentence i with at most $n + 1$ occurrences of connectives, any sentences ψ and ψ' such that ψ is a *proper* subsentence of i (the case in which ψ is i is trivial) and ψ is logically equivalent to ψ' , and any sentence i' that results from replacing in i at least one occurrence of ψ with ψ' (to abbreviate, say that i' is $i(\psi/\psi')$). To prove that i is logically equivalent to i' , there are seven cases to consider (the same as the seven cases in §A.1.1).

Case 1: i is $\ulcorner!p\urcorner$. Then ψ is a subsentence of p , and i' is $\ulcorner!p'\urcorner$, where p' is $p(\psi/\psi')$ and thus is logically equivalent to p (since the lemma holds for declarative sentences). It follows, similarly to the base case, that i is logically equivalent to i' .

Case 2: i is $\ulcorner\sim j\urcorner$. Then ψ is a subsentence of j , and i' is $\ulcorner\sim j'\urcorner$, where j' is $j(\psi/\psi')$ and thus, by the induction hypothesis, is logically equivalent to j (because j has at most n occurrences of connectives). It follows that i is logically equivalent to i' : for any interpretation m , i is satisfied on m exactly if j is violated on m (by C9), and thus exactly if j' is violated on m (by the logical equivalence of j and j'), and thus exactly if i' is satisfied on m (and similarly for violation on m).

Case 3: i is $\ulcorner j \& k \urcorner$. Then ψ is a subsentence of j or of k . Suppose it is of j (if it is of k , the proof proceeds similarly). Then i' is $\ulcorner j' \& k \urcorner$, where j' is $j(\psi/\psi')$ and thus, by the induction hypothesis, is logically equivalent to j . It follows that i is logically equivalent to i' : for any interpretation m , i is satisfied on m exactly if either (i) both j and k are satisfied on m or (ii) one of j and k is satisfied on m and the other one is neither satisfied nor violated on m (by C10), and thus exactly if either (i) both j' and k are satisfied on m or (ii) one of j' and k is satisfied on m and the other one is neither satisfied nor violated on m , and thus exactly if i' is satisfied on m (and similarly for violation on m).

The proof proceeds similarly in the remaining four cases, which I omit for the sake of brevity.

A.1.3. Proof of the Equivalence Lemma

The lemma provides necessary and sufficient conditions for strong and for weak semantic entailment. The proof has four parts, and is similar to the proof in Appendix A of Vranas 2011.

First part: Sufficient condition for strong semantic entailment. If i is a contradiction, then (by Definition 1) no declarative sentence strongly supports i on any interpretation, and then (by Definition 2) i strongly semantically entails j . If both (a) i is satisfied on every interpretation on which

j is satisfied and (b) i is violated on every interpretation on which j is violated, take any interpretation m and any declarative sentence p . Suppose that (1) p strongly supports i on m . Then (2) p is true on m (by Definition 1) and (3) j is not a contradiction (because, by Definition 1, i is not a contradiction, so on some interpretation i is not violated, and then by (b) on some interpretation j is not violated either). Moreover, (4) for any declarative sentences q and r that are not both contradictions, if (i) j is satisfied on every interpretation on which q is true and (ii) j is violated on every interpretation on which r is true, then p favors q over r on m (by (1) and Definition 1, because (by (i) and (a)) i is satisfied on every interpretation on which q is true and (by (ii) and (b)) i is violated on every interpretation on which r is true). By (2), (3), (4), and Definition 1, p strongly supports j on m , so i strongly semantically entails j .

Second part: Necessary condition for strong semantic entailment. Take a sentence s that is true on all and only those interpretations on which i is satisfied, a sentence v that is true on all and only those interpretations on which i is violated, and sentences s' and v' that satisfy the corresponding conditions with respect to j .³² Suppose, for reductio, that (1) i strongly semantically entails j but (2) i is not a contradiction and (3) it is not the case that both (a) i is satisfied on every interpretation on which j is satisfied and (b) i is violated on every interpretation on which j is violated. Consider an interpretation m whose first coordinate is $\{p\}$ (so (4) p is true on m), where p is a sentence letter, and whose second coordinate is the set of ordered triples $\langle p', q, r \rangle$ such that (i) p' is any declarative sentence logically equivalent to p , (ii) q semantically entails s (i.e., i is satisfied on every interpretation on which q is true), (iii) r semantically entails v (i.e., i is violated on every interpretation on which r is true), and (iv) q and r are not both contradictions.³³ By (2), (4), the definition of the second coordinate of m , and Definition 1, p strongly supports i on m . Then, by (1) and Definition 2, (5) p also strongly supports j on m . Let q be $\lceil s' \& \sim s \rceil$ and r be $\lceil v' \& \sim v \rceil$. By (3), q and r are not both contradictions. Moreover, j is satisfied on every interpretation on which q is true, and j is violated on every interpretation on which r is true. Then, by (5) and Definition 1, p favors q over r on m ; i.e., $\langle p, q, r \rangle$ is in the second coordinate of m . By (ii), $\lceil s' \& \sim s \rceil$ semantically entails s , so (there is no interpretation on which $\lceil (s' \& \sim s) \& \sim s \rceil$ is true, and thus) s' semantically entails s ; i.e., (a) holds. Similarly, by (iii), $\lceil v' \& \sim v \rceil$ semantically entails v , so v' semantically entails v ; i.e., (b) holds. But (a) and (b) together contradict (3), and the reductio is complete.

Third part: Sufficient condition for weak semantic entailment. Suppose (a) j is avoided on every interpretation on which i is avoided and (b) i is violated on every interpretation on which j is violated. Take any interpretation m and any declarative sentence p that weakly supports i on m . By Definition 1, (1) p strongly supports on m some imperative sentence i^* such that (i) i is satisfied on every interpretation on which i^* is satisfied and (ii) i is avoided on all and only those interpretations on which i^* is avoided. Let k be $\lceil (s' \vee v') \rightarrow !(s^* \& s) \rceil$, where s' and v' are as in the second part of the proof and s^* is a sentence that is true on all and only those interpretations on

³² To see that such sentences exist, note that there are declarative sentences p and q such that i and $\lceil p \rightarrow !q \rceil$ are replacement interderivable (by the Canonical Form Lemma) and thus logically equivalent (by the Replacement Lemma and the fact—see note 34—that all replacement rules are based on logical equivalences). Then i is satisfied on all and only those interpretations on which $\lceil p \rightarrow !q \rceil$ is satisfied, namely (by C12 and C8) on which both p and q are true, so take s to be $\lceil p \& q \rceil$. Similarly, take v to be $\lceil p \& \sim q \rceil$.

³³ The second coordinate of m satisfies the asymmetry condition (§2.2): if one supposes for reductio that $\langle p', q, r \rangle$ and $\langle p', r, q \rangle$ are both in the second coordinate, then one gets that q and r are both contradictions (contradicting (iv)): q is a contradiction because it semantically entails both s and v (and there is no interpretation on which i is both satisfied and violated), and similarly for r . The intensionality condition (§2.2) is also satisfied.

which i^* is satisfied. Then (2) k is satisfied only on interpretations on which $\lceil s^* \ \& \ s \rceil$ is true, so (3) i^* is satisfied on every interpretation on which k is satisfied. Moreover, (4) i^* is violated on every interpretation on which k is violated (as one can show by using (a), (b), (i), and (ii); see Vranas 2011: 436 n. 68). By (3), (4), and the first part of the lemma, (5) i^* strongly semantically entails k . By (1), (5), and Definition 2, (6) p strongly supports k on m . But (7) j is satisfied on every interpretation on which k is satisfied (by (2)), and (8) j is avoided on all and only those interpretations on which k is avoided (because k is avoided on all and only those interpretations on which $\lceil s' \vee v' \rceil$ is false). By (6), (7), (8), and Definition 1, p weakly supports j on m , so i weakly semantically entails j .

Fourth part: Necessary condition for weak semantic entailment. Suppose, for reductio, that (1) i weakly semantically entails j but (2) it is not the case that both (a) j is avoided on every interpretation on which i is avoided and (b) i is violated on every interpretation on which j is violated. By (2), i is not a contradiction (because i is not violated on every interpretation: either (a) is false, and then on some interpretation i is avoided and thus not violated, or (b) is false, and then on some interpretation i is not violated). Consider an interpretation m defined as in the second part of the proof. As in that part, a sentence letter p strongly supports i on m , so p also weakly supports i on m . Then, by (1) and Definition 2, p also weakly supports j on m . Then, by Definition 1, (3) p strongly supports on m some imperative sentence i^* such that (i) j is satisfied on every interpretation on which i^* is satisfied and (ii) j is avoided on all and only those interpretations on which i^* is avoided. By (2), j is not avoided on every interpretation (because either (a) is false, and then on some interpretation j is not avoided, or (b) is false, and then on some interpretation j is violated and thus not avoided), so (by (ii)) i^* is not avoided on every interpretation and thus s^* and v^* are not both contradictions (where s^* is as in the third part of the proof and v^* is a sentence that is true on all and only those interpretations on which i^* is violated). Then, by (3) and Definition 1, p favors s^* over v^* on m , and by the definition of m in the second part of the proof, (4) s^* semantically entails s and (5) v^* semantically entails v . But then (a) holds: on every interpretation on which j (and thus, by (ii), i^*) is not avoided, i is not avoided either (because either i^* is satisfied, and then by (4) i is also satisfied and thus not avoided, or i^* is violated, and then by (5) i is also violated and thus not avoided). Moreover, (b) holds: on every interpretation on which j is violated (and thus neither satisfied nor avoided), i is also violated (because i^* is not satisfied, by (i) and the fact that j is not satisfied, and i^* is not avoided, by (ii) and the fact that j is not avoided, so i^* is violated and by (5) so is i). But (a) and (b) together contradict (2), and the reductio is complete.

A.1.4. Proof of the Soundness and Completeness Theorem

Proof of soundness. Take any non-empty finite set Γ of imperative sentences and any imperative sentences i and j . For brevity, I prove together the three claims that, if j is (1) replacement derivable from i , (2) strongly derivable from Γ , or (3) derivable from Γ , then, respectively, (1) i is logically equivalent to j , (2) Γ strongly semantically entails j , or (3) Γ weakly semantically entails j . (I assume that the analog of (1) for declarative sentences holds, since it corresponds to a well-known result from classical sentential logic.) The proof is by induction on the number of lines of a replacement derivation, strong derivation, or derivation. For the base step, suppose there is a one-line (case 1) replacement derivation of j from i , (case 2) strong derivation of j from Γ , or (case 3) derivation of j from Γ . In case 1, i is the same sentence as j and thus is logically equivalent to j . In case 2, j is a conjunction of all members of Γ and thus (by Definition 2) Γ strongly semantically entails j . In case 3, j is (a member or) a conjunction of members of Γ ; so, if j is not a

conjunction of *all* members of Γ (if it is, the proof proceeds as in case 2), there is a conjunction k of the remaining members of Γ , and $\lceil j \ \& \ k \rceil$ is a conjunction of all members of Γ . Then Γ weakly semantically entails j because, by Definition 2, Γ weakly semantically entails $\lceil j \ \& \ k \rceil$, and by the Equivalence Lemma, $\lceil j \ \& \ k \rceil$ weakly semantically entails j : by C10 (§2.2), (a) j is avoided on every interpretation on which $\lceil j \ \& \ k \rceil$ is avoided and (b) $\lceil j \ \& \ k \rceil$ is violated on every interpretation on which j is violated.

For the inductive step, take any non-zero natural number n and suppose (*induction hypothesis*):

(Case 1) If there is a replacement derivation with at most n lines of j from i , then i is logically equivalent to j .

(Case 2) If there is a strong derivation with at most n lines of j from Γ , then Γ strongly semantically entails j .

(Case 3) If there is a derivation with at most n lines of j from Γ , then Γ weakly semantically entails j .

To complete the proof, take any (case 1) replacement derivation with at most $n + 1$ lines of j from i , (case 2) strong derivation with at most $n + 1$ lines of j from Γ , or (case 3) derivation with at most $n + 1$ lines of j from Γ . Suppose that j is *not* (case 1) the same sentence as i , (case 2) a conjunction of all members of Γ , or (case 3) a conjunction of members of Γ (if it is, the proof proceeds as in the base step).

In case 1, j can be obtained from an n' -th line k ($n' \leq n$) by applying once a replacement rule. By the induction hypothesis and the fact that the sequence of the first n' lines of the replacement derivation of j from i is a replacement derivation with at most n lines of k from i , (1) i is logically equivalent to k . By the Replacement Lemma and the fact that all replacement rules are based on logical equivalences,³⁴ (2) k is logically equivalent to j . By (1), (2), and the transitivity of logical equivalence (which follows from its definition in §2.2), i is logically equivalent to j .

In case 2, j can be obtained from an n' -th line k ($n' \leq n$) by applying once SA, ECQ, or a replacement rule. By the induction hypothesis and the fact that the sequence of the first n' lines of the strong derivation of j from Γ is a strong derivation with at most n lines of k from Γ , (1) Γ strongly semantically entails k . By using the Equivalence Lemma and the Replacement Lemma, one can show that SA, ECQ, and all replacement rules always correspond to strongly semantically valid arguments,³⁵ so (2) k strongly semantically entails j . By (1), (2), and the transitivity of strong semantic entailment (which follows from Definition 2), Γ strongly semantically entails j .

In case 3, j can be obtained from one, two, or (for UDD) three previous lines by applying once a replacement or an inference rule. For any n' -th line k among those that can be used to obtain j , $n' \leq n$, so by the induction hypothesis and the fact that the sequence of the first n' lines of the deri-

³⁴ One can show that all replacement rules are based on logical equivalences by using C1-C15 (§2.2). For example, take the imperative part of EX. For any interpretation m : by C12, $\lceil p \rightarrow (q \rightarrow i) \rceil$ is satisfied on m exactly if both p is true on m and $\lceil q \rightarrow i \rceil$ is satisfied on m ; i.e., by C12, exactly if both (a) p is true on m and (b) both q is true on m and i is satisfied on m ; i.e., by C3, exactly if both $\lceil p \ \& \ q \rceil$ is true on m and i is satisfied on m ; i.e., by C12, exactly if $\lceil (p \ \& \ q) \rightarrow i \rceil$ is satisfied on m (and similarly for violation on m).

³⁵ Concerning replacement rules, reason as in case 1 and recall that i strongly semantically entails j if i is logically equivalent to j . Concerning inference rules, take for example the first part of SA. (a) On every interpretation on which $\lceil p \rightarrow i \rceil$ is satisfied, both p is true and i is satisfied (by C12), so i is satisfied. (b) On every interpretation on which $\lceil p \rightarrow i \rceil$ is violated, both p is true and i is violated (by C12), so i is violated. By (a), (b), and the Equivalence Lemma, i strongly semantically entails $\lceil p \rightarrow i \rceil$. One can show similarly that all replacement and inference rules always correspond to *weakly* semantically valid arguments.

vation of j from Γ is a derivation with at most n lines of k from Γ , Γ weakly semantically entails k . By using the Equivalence Lemma and the Replacement Lemma, one can show that then (1) Γ weakly semantically entails any conjunction of all lines that can be used to obtain j ,³⁶ and that all replacement and inference rules always correspond to weakly semantically valid arguments (see note 35), so (2) any conjunction of all lines that can be used to obtain j weakly semantically entails j . By (1), (2), and the transitivity of weak semantic entailment (which follows from Definition 2), Γ weakly semantically entails j .

Proof of completeness. I prove first completeness concerning logical equivalence. Take any logically equivalent sentences ϕ and ψ . I assume that ϕ and ψ are replacement interderivable if they are declarative sentences (since this case corresponds to a result from classical logic; I sketch a proof in a note³⁷), so suppose ϕ and ψ are imperative sentences i and j . By the Canonical Form Lemma, there are declarative sentences p , q , p' , and q' such that (1) i and $\lceil p \rightarrow !q \rceil$ are replacement interderivable and thus (by soundness) logically equivalent, and (2) j and $\lceil p' \rightarrow !q' \rceil$ are replacement interderivable and thus logically equivalent. Then (3) $\lceil p \rightarrow !q \rceil$ and $\lceil p' \rightarrow !q' \rceil$ are logically equivalent. It follows that p and p' are logically equivalent (and thus (4) p and p' are replacement interderivable, since they are declarative sentences): on any interpretation m , p is true exactly if either both p is true and q is true or both p is true and q is false, so exactly if either $\lceil p \rightarrow !q \rceil$ is satisfied or $\lceil p \rightarrow !q \rceil$ is violated, so (by (3)) exactly if either $\lceil p' \rightarrow !q' \rceil$ is satisfied or $\lceil p' \rightarrow !q' \rceil$ is violated, so exactly if either both p' is true and q' is true or both p' is true and q' is false, so exactly if p' is true. One can show similarly that $\lceil p \& q \rceil$ and $\lceil p' \& q' \rceil$ are logically equivalent, so (5) $\lceil p \& q \rceil$ and $\lceil p' \& q' \rceil$ are replacement interderivable. To conclude: i is replacement interderivable, by (1), with $\lceil p \rightarrow !q \rceil$, and thus also, by AB, with $\lceil p \rightarrow !(p \& q) \rceil$, and thus also, by (4), with $\lceil p' \rightarrow !(p \& q) \rceil$, and thus also, by (5), with $\lceil p' \rightarrow !(p' \& q') \rceil$, and thus also, by AB, with $\lceil p' \rightarrow !q' \rceil$, and thus finally, by (2), with j .

³⁶ For example, suppose two lines, k and k' , can be used to obtain j . Then Γ weakly semantically entails both k and k' , so any conjunction l of all members of Γ weakly semantically entails both k and k' . By the Equivalence Lemma, (a) both k and k' are avoided on every interpretation on which l is avoided, and (b) l is violated on every interpretation on which either k or k' is violated. By C10, $\lceil k \& k' \rceil$ is avoided on every interpretation on which both k and k' are avoided, so—by (a)—(1) $\lceil k \& k' \rceil$ is avoided on every interpretation on which l is avoided. By C10, either k or k' is violated on every interpretation on which $\lceil k \& k' \rceil$ is violated, so—by (b)—(2) l is violated on every interpretation on which $\lceil k \& k' \rceil$ is violated. By (1), (2), and the Equivalence Lemma, l weakly semantically entails $\lceil k \& k' \rceil$, and thus so does Γ .

³⁷ Take any logically equivalent declarative sentences p and q , let Γ' be the set of sentence letters that occur in p or in q , and let n be the number of members of Γ' . If p and q are not contradictions, there are replacement derivations (in which only MI and the declarative parts of DN, IP, CO, AS, DI, ME, DM, and TC may be applied) of declarative sentences p' and q' from p and q respectively such that each of p' and q' is in disjunctive Boolean normal form; i.e., it is a disjunction (defining disjunctions similarly to conjunctions; see note 8) of n' ($\leq 2^n$) distinct sentences such that (1) all n members of Γ' occur, and in the same order (for both p' and q'), in each of the n' sentences, and (2) each of the n' sentences begins with a string of $n - 1$ left parentheses and is a conjunction of n sentences each of which either is a member of Γ' or is $\lceil \sim r \rceil$ for some member r of Γ' . (For example, if $\Gamma' = \{A, B, C\}$, then $\lceil ((A \& \sim B) \& C) \vee ((\sim A \& \sim B) \& \sim C) \vee ((A \& B) \& C) \rceil$ is in disjunctive Boolean normal form.) By soundness, both p' is logically equivalent to p and q' is logically equivalent to q ; so p' is logically equivalent to q' . But then p' and q' have exactly the same disjuncts (and thus are replacement interderivable by applying only CO and AS): if some disjunct of p' is not a disjunct of q' , then on some (indeed, on every) interpretation on which that disjunct (and thus p') is true every disjunct of q' (and thus q' itself) is false, and then p' is not logically equivalent to q' . To conclude: p is replacement interderivable with p' , and thus also with q' , and thus also with q . (If p and q are contradictions, a similar proof works by considering *conjunctive* Boolean normal forms.)

I prove next completeness concerning strong and weak semantic validity. Take any imperative sentence j , any finite non-empty set Γ of imperative sentences, and any conjunction i of all members of Γ . By the Canonical Form Lemma, there are declarative sentences p , q , p' , and q' such that (1) i is replacement interderivable with $\lceil p \rightarrow !q \rceil$ and (2) j is replacement interderivable with $\lceil p' \rightarrow !q \rceil$. Then, by soundness, (3) i is logically equivalent to $\lceil p \rightarrow !q \rceil$ and (4) j is logically equivalent to $\lceil p' \rightarrow !q \rceil$. *Case 1: Γ strongly semantically entails j .* Then (5) i strongly semantically entails j . *Case 1a: i is a contradiction.* Then, for any declarative sentence r , i and $\lceil!(r \& \sim r)\rceil$ are logically equivalent and thus (by completeness for logical equivalence) replacement interderivable, so there is a (replacement, and thus) strong derivation of $\lceil!(r \& \sim r)\rceil$ from i . Then there is also a strong derivation of j from i (and thus from Γ), since j can be obtained from $\lceil!(r \& \sim r)\rceil$ by ECQ. *Case 1b: i is not a contradiction.* Then, by (5) and the Equivalence Lemma, (6) i is satisfied on every interpretation on which j is satisfied and (7) i is violated on every interpretation on which j is violated. By (3), (4), and (6): (8) $\lceil p' \& q \rceil$ semantically entails $\lceil p \& q \rceil$. By (3), (4), and (7): (9) $\lceil p' \& \sim q \rceil$ semantically entails $\lceil p \& \sim q \rceil$. By using (8) and (9), one can show that (10) p' semantically entails p and that $\lceil p' \& (p \& q) \rceil$ and $\lceil p' \& q \rceil$ are logically equivalent and thus (11) replacement interderivable. To conclude: there is a strong derivation from Γ of i , and thus also, by (1), of $\lceil p \rightarrow !q \rceil$, and thus also, by AB, of $\lceil p \rightarrow !(p \& q) \rceil$, and thus also, by (10) and SA, of $\lceil p' \rightarrow !(p \& q) \rceil$, and thus also, by AB, of $\lceil p' \rightarrow !(p' \& (p \& q)) \rceil$, and thus also, by (11), of $\lceil p' \rightarrow !(p' \& q) \rceil$, and thus also, by AB, of $\lceil p' \rightarrow !q \rceil$, and thus finally, by (2), of j . *Case 2: Γ weakly semantically entails j .* Then i weakly semantically entails j . By the Equivalence Lemma, (12) j is avoided on every interpretation on which i is avoided and (13) i is violated on every interpretation on which j is violated. By (3), (4), and (12): (14) p' semantically entails p . By (3), (4), and (13): (15) $\lceil p' \& \sim q \rceil$ semantically entails $\lceil p \& \sim q \rceil$. By using (15), one can show that (16) $\lceil p' \& q \rceil$ semantically entails q' . To conclude: there is a derivation from Γ of i , and thus also, by (1), of $\lceil p \rightarrow !q \rceil$, and thus also, by (14) and SA, of $\lceil p' \rightarrow !q \rceil$, and thus also, by AB, of $\lceil p' \rightarrow !(p' \& q) \rceil$, and thus also, by (16) and WC, of $\lceil p' \rightarrow !q \rceil$, and thus finally, by (2), of j .³⁸

A.2. Sentential modal imperative logic (SMIL)

My main goal here is to prove the following theorem:

SOUNDNESS AND COMPLETENESS THEOREM (FOR SMIL).

For any constraint c that entails ρ (and, for mixed declarative arguments, also entails ϵ), an argument is (1) strongly semantically c -valid exactly if it is strongly syntactically c -valid, and is (2) weakly semantically c -valid exactly if it is weakly syntactically c -valid.³⁹

³⁸ From the proof of completeness (including note 37) and the proof of the Canonical Form Lemma, one can see that: (1) the set consisting of (a) the replacement rules CO, UN, IC, and ID, (b) the imperative parts of TA, NC, EX, TR, ME, and AB, and (c) the declarative parts of DN, IP, AS, DI, DM, and TC is complete with respect to logical equivalence; (2) the set consisting of the members of the set in (1) and of the inference rules SA and ECQ is complete with respect to strong semantic entailment; and (3) the set consisting of the members of the set in (1) and of the inference rules SA and WC is complete with respect to weak semantic entailment. Alternative sets of (parts of) rules that are complete with respect to strong and to weak semantic entailment can be obtained by recalling that, as explained in §2.4.2, both SA and WC are redundant given ICE, UC, the imperative parts of TA, DI, and EX, and a set of replacement rules complete with respect to logical equivalence.

³⁹ A *soundness* theorem concerning logical (c -)equivalence also holds: for any c , sentences ϕ and ψ are logically (c -)equivalent if they are *replacement (c -)interderivable* (i.e., there are replacement c -derivations of ψ from ϕ and of ϕ from ψ). To obtain a *completeness* theorem concerning logical (c -)equivalence, one would need to (re)define replacement (c -)interderivability so as to also allow applying (1) what I call the “Modal Negation” replacement rule (namely the replacement rule which is based on the general logical equivalences between $\lceil \sim \Box p \rceil$ and $\lceil \Diamond \sim p \rceil$ and

Since, as I explained in §3.4.2, every strongly *syntactically* c -valid argument is also weakly syntactically c -valid, an immediate corollary of the Soundness and Completeness Theorem is that, for any c that entails ρ (and, for mixed declarative arguments, also entails ε), every strongly *semantically* c -valid argument is also weakly semantically c -valid. In what follows, I use without proof the Canonical Form Lemma (for SMIL) and the Replacement Lemma (for SMIL); these two lemmata have formulations identical to the formulations of the corresponding lemmata for SPIL in §A.1 (although here one quantifies over sentences of the language of SMIL, not just over sentences of the language of SPIL), and have proofs very similar to the proofs of the corresponding lemmata for SPIL in §A.1.1 and §A.1.2. Before I prove the theorem, I prove the following lemma:

EQUIVALENCE LEMMA (FOR SMIL). Take any declarative sentences t and t' , any imperative sentences i and i' , and any satisfaction sentences s and s' , violation sentences v and v' , and avoidance sentences a and a' of i and i' respectively.

- (1) Concerning pure declarative arguments: For any c that entails ρ , the argument from t to t' is semantically c -valid exactly if, for any c -interpretation m and any world w of m , t' is true at w on m if t is true at w on m .
- (2) Concerning cross-species imperative arguments: For any c that entails ρ , the argument from t to i is strongly semantically c -valid exactly if it is weakly semantically c -valid and also exactly if t semantically c -entails $\lceil \Box a \rceil$.
- (3) Concerning mixed-premise declarative arguments: For any c that entails $\rho\varepsilon$, the argument from t and i to t' is strongly semantically c -valid exactly if it is weakly semantically c -valid and also exactly if $\lceil t \ \& \ \Diamond \sim v \rceil$ semantically c -entails t' .
- (4) Concerning mixed-premise imperative arguments: For any c that entails ρ , the argument from t and i to i' is (a) *strongly* semantically c -valid exactly if $\lceil t \ \& \ \Diamond \sim v \rceil$ semantically c -entails both $\lceil \Box(s' \rightarrow s) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$, and (b) is *weakly* semantically c -valid exactly if t semantically c -entails both $\lceil \Box(a \rightarrow a') \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$.
- (5) Concerning cross-species declarative arguments: For any c that entails $\rho\varepsilon$, the argument from i to t' is strongly semantically c -valid exactly if it is weakly semantically c -valid and also exactly if $\lceil \Diamond \sim v \rceil$ semantically c -entails t' .
- (6) Concerning pure imperative arguments: For any c that entails ρ , the argument from i to i' is (a) *strongly* semantically c -valid exactly if $\lceil \Diamond \sim v \rceil$ semantically c -entails both $\lceil \Box(s' \rightarrow s) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$, and (b) is *weakly* semantically c -valid exactly if both a semantically c -entails a' and v' semantically c -entails v .

Although the Equivalence Lemma applies directly only to arguments with a *single* declarative premise or a *single* imperative premise (or both), directly or indirectly it applies to *every* argument: by Definition 4c, and given the intensionality condition and the logical equivalence of any two conjunctions of all declarative premises or any two conjunctions of all imperative premises, one can replace multiple declarative premises with any conjunction of all of them and multiple imperative premises with any conjunction of all of them without affecting strong or weak semantic c -validity. A first corollary of the Equivalence Lemma is that a cross-species imperative argument (for any c that entails ρ ; see part 2 of the lemma) and a mixed—i.e., either mixed-premise or cross-species—declarative argument (for any c that entails $\rho\varepsilon$ or, equivalently, $\rho\sigma\tau$;

between $\lceil \sim \Diamond p \rceil$ and $\lceil \Box \sim p \rceil$) and (2) replacement rules specific to c (e.g., for $\rho\tau$, the replacement rule which is based on the general logical $\rho\tau$ -equivalence between $\lceil \Box \Box p \rceil$ and $\lceil \Box p \rceil$).

see parts 3 and 5 of the lemma) is strongly semantically c -valid *exactly if* it is weakly semantically c -valid. A second corollary of the Equivalence Lemma is that for any *mixed* argument there is a *pure declarative* argument which is strongly semantically c -valid exactly if so is the mixed argument, and similarly for weak semantic c -validity (for any c that entails ρ for mixed imperative arguments, and any c that entails $\rho\epsilon$ for mixed declarative arguments; see parts 2, 3, 4, and 5 of the lemma). A third corollary of the Equivalence Lemma is that a *pure* argument without modal operators is (strongly or weakly) semantically c -valid in SMIL exactly if it is (strongly or weakly) semantically valid in SPIL. Concerning (1) pure declarative arguments and (2) *weak* semantic (c -)validity for pure imperative arguments, the corollary can be proved by comparing respectively (1) part 1 of the lemma with what I say about pure declarative arguments in §2.3 and (2) part 6b of the lemma with the Equivalence Lemma for SPIL in §A.1; concerning (3) *strong* semantic (c -)validity for pure imperative arguments, I prove the corollary in a note.⁴⁰ Finally, I state without proof five further corollaries of the Equivalence Lemma in a note.⁴¹

In §A.2.1 I prove the Equivalence Lemma, and in §A.2.2 I prove the theorem.

A.2.1. Proof of the Equivalence Lemma

⁴⁰ Comparing part 6a of the lemma with the Equivalence Lemma for SPIL in §A.1, one can see that it is enough to prove that, for any declarative sentences p and q that contain no modal operators, $\lceil \Diamond p \rceil$ semantically c -entails $\lceil \Box q \rceil$ if and only if either p is a contradiction or q is a tautology (because then $\lceil \Diamond \sim v \rceil$ semantically c -entails $\lceil \Box((s' \rightarrow s) \& (v' \rightarrow v)) \rceil$ exactly if either $\lceil \sim v \rceil$ is a contradiction or $\lceil (s' \rightarrow s) \& (v' \rightarrow v) \rceil$ is a tautology). The “if” part holds because, if p is a contradiction or q is a tautology, then $\lceil \Diamond p \rceil$ is a contradiction or $\lceil \Box q \rceil$ is a tautology. To prove the (contrapositive of the) “only if” part, suppose that p is not a contradiction and q is not a tautology. Then there are interpretations m and m' , and worlds w of m and w' of m' , such that p is true at w on m and q is false at w' on m' . Consider a c -interpretation m^* whose first coordinate is a set $\{w_0, w_1, w_2\}$, whose second coordinate is an accessibility relation such that c is satisfied and both w_1 and w_2 are accessible from w_0 , and whose third coordinate is a function that (1) assigns to w_1 the set of sentence letters that the third coordinate of m assigns to w and (2) assigns to w_2 the set of sentence letters that the third coordinate of m' assigns to w' . (It does not matter what the fourth coordinate is.) Because p and q contain no modal operators, whether they are true at a world on an interpretation depends only on which sentence letters are true at that world on the interpretation, so p is true at w_1 on m^* and q is false at w_2 on m^* . Then $\lceil \Diamond p \rceil$ is true and $\lceil \Box q \rceil$ is false at w_0 on m^* , so $\lceil \Diamond p \rceil$ does not semantically c -entail $\lceil \Box q \rceil$. (The result does not hold if c entails that at most one world is accessible from any given world, because then, for example, $\lceil \Diamond M \rceil$ semantically c -entails $\lceil \Box M \rceil$.)

⁴¹ (1) For any sentences ϕ and ψ that are either both declarative or both imperative, and for any c that entails ρ , the arguments from ϕ to ψ and from ψ to ϕ are both weakly semantically c -valid exactly if they are both strongly semantically c -valid and also exactly if ϕ and ψ are logically c -equivalent (see parts 1 and 6 of the lemma). (2) For any c that entails $\rho\epsilon$, some declarative sentence and some imperative sentence strongly and weakly semantically c -entail each other; for example, the sentences $\lceil M \vee \sim M \rceil$ and $\lceil (M \& \sim M) \rightarrow !P \rceil$, and also the sentences $\lceil \Box \sim \Box M \rceil$ and $\lceil \Box M \rightarrow !\sim \Box M \rceil$ (see parts 2 and 5 of the lemma). Note, however, that $\lceil (M \& \sim M) \rightarrow !P \rceil$ is *not* an imperative tautology and is *not* logically c -equivalent to $\lceil M \vee \sim M \rceil$. (3) For any c that entails ρ , the argument from $\lceil t \rightarrow i \rceil$ to $\lceil t \rightarrow i' \rceil$ is (a) weakly semantically c -valid if so is the argument from t and i to i' , and (b) is weakly or strongly semantically c -valid only if so is the argument from $\lceil \Box t \rceil$ and i to i' . (4) For any c that entails $\rho\tau$, the argument from $\lceil \Box t \rightarrow i \rceil$ to $\lceil \Box t \rightarrow i' \rceil$ is weakly semantically c -valid exactly if—and is strongly semantically c -valid only if—so is the argument from $\lceil \Box t \rceil$ and i to i' . (5) For any c that entails ρ , the argument from i to $\lceil t \rightarrow i' \rceil$ is weakly semantically c -valid if so is the argument from i and t to i' . Corollaries (3)-(5) relate pure imperative arguments to mixed-premise imperative arguments. Corollary (5) could be used to introduce a restricted version of Conditional Proof: if one modified my definition of a c -derivation so as to allow the introduction of provisional assumptions in subderivations, to c -derive $\lceil t \rightarrow i' \rceil$ from *only imperative premises* one could assume t and c -derive i' . “Unrestricted Conditional Proof” fails, however: to c -derive $\lceil t' \rightarrow i' \rceil$ from t and i it would *not* be enough to assume t' and c -derive i' . For example, if t is $\lceil D \rightarrow \Box(A \& (B \rightarrow C)) \rceil$, i is $\lceil A \rightarrow !B \rceil$, t' is $\lceil D \rceil$ and i' is $\lceil !C \rceil$, the argument from $\lceil t \& t' \rceil$ and i to i' is weakly semantically ρ -valid but the argument from t and i to $\lceil t' \rightarrow i' \rceil$ is not (as one can show by using the Equivalence Lemma).

(1A) *Pure declarative arguments: sufficient condition for semantic c-entailment.* Suppose that (1) for any c -interpretation m^* and any world w^* of m^* , t' is true at w^* on m^* if t is true at w^* on m^* . Take any c -interpretation m , any world w of m , and any declarative sentence p that guarantees t at w on m ; i.e., (2) p is true at w on m and (3) $\lceil \Box(p \rightarrow t) \rceil$ is true at w on m . By (3) and C16* (see §3.2), for any world w' of m accessible from w , $\lceil p \rightarrow t \rceil$ is true at w' on m , and thus (by (1) and C2*-C5*) $\lceil p \rightarrow t \rceil$ is true at w' on m . Then, by C16*, (4) $\lceil \Box(p \rightarrow t) \rceil$ is true at w on m . By (2) and (4), p guarantees t' at w on m . Then, by Definition 4c, t semantically c -entails t' .

(1B) *Pure declarative arguments: necessary condition for semantic c-entailment.* Suppose that (1) t semantically c -entails t' . Take any c -interpretation m and any world w of m . Suppose that (2) t is true at w on m . By C2*-C5*, for any world w' of m accessible from w , $\lceil t \rightarrow t' \rceil$ is true at w on m . Then, by C16*, (3) $\lceil \Box(t \rightarrow t') \rceil$ is true at w on m . By (2) and (3), t guarantees t' at w on m . Then, by (1) and Definition 4c, t guarantees t' at w on m , so (4) $\lceil \Box(t \rightarrow t') \rceil$ is true at w on m . Given that c entails ρ , w is accessible from w ; so, by (4) and C16*, (5) $\lceil t \rightarrow t' \rceil$ is true at w on m . By (2), (5), and C2*-C5*, t' is true at w on m .

(2A) *Cross-species imperative arguments: sufficient condition for strong and for weak semantic c-entailment.* Suppose that (1) t semantically c -entails $\lceil \Box a \rceil$. Take any c -interpretation m , any world w of m , and any declarative sentence p . Suppose that p guarantees t at w on m . Then, by (1) and Definition 4c, p guarantees $\lceil \Box a \rceil$ at w on m ; i.e., (2) p is true at w on m and (3) $\lceil \Box(p \rightarrow \Box a) \rceil$ is true at w on m . By (3) and C16*, and given that c entails ρ , $\lceil p \rightarrow \Box a \rceil$ is true at w on m , and thus (by (2) and C2*-C5*) so is $\lceil \Box a \rceil$. Then, by C16*, for any world w' of m accessible from w , a is true at w' on m , and thus (by C15* and C2*) so are $\lceil \sim s \rceil$ and $\lceil \sim v \rceil$. Then, by C16*, (4) both $\lceil \Box \sim s \rceil$ and $\lceil \Box \sim v \rceil$ are true at w on m . But then, given that c entails ρ , (5) $\lceil \Box v \rceil$ is not true at w on m . By (4), C16*, and C2*-C5*: (6) for any declarative sentences q and r , if both $\lceil \Box(q \rightarrow s) \rceil$ and $\lceil \Box(r \rightarrow v) \rceil$ are true at w on m , then both $\lceil \Box \sim q \rceil$ and $\lceil \Box \sim r \rceil$ are true at w on m . By (2), (5), (6), and Definition 3, p strongly and weakly supports i at w on m . Then, by Definition 4c, t strongly and weakly semantically c -entails i .

(2B) *Cross-species imperative arguments: necessary condition for strong and for weak semantic c-entailment.* Suppose, for reductio, that (1) t weakly semantically c -entails i but (2) t does not semantically c -entail $\lceil \Box a \rceil$. (By Definition 3, if p strongly supports i at w on m , then p also weakly supports i at w on m ; so, if t strongly semantically c -entails i , then (1) also holds by Definition 4c, and the proof proceeds as below.) By (2) and the first part of the lemma, there is a c -interpretation m and a world w of m such that (3) t is true at w on m but (4) $\lceil \Box a \rceil$ is not true at w on m . Consider an interpretation m' that (5) has the same first three coordinates as m but (6) has as its fourth coordinate a function that assigns to w the empty favoring relation (i.e., the empty set). By C1*-C17*, whether a sentence is true at a world on an interpretation does not depend on the fourth coordinate of the interpretation. Then, by (3), (4), and (5): (7) t is true at w on m' but (8) $\lceil \Box a \rceil$ is not true at w on m' . By (7), t guarantees t at w on m' (because $\lceil \Box(t \rightarrow t) \rceil$ is true at w on m'). Then, by (1) and Definition 4c, t weakly supports i at w on m' . Then, by Definition 3, there is an imperative sentence i' such that (9) t strongly supports i' at w on m' and (10) $\lceil \Box(a \leftrightarrow a') \rceil$ is true at w on m' . By (8) and (10), $\lceil \Box a \rceil$ is not true at w on m' . Then there is a world w' of m' , accessible from w , such that a' is not true at w' on m' , and thus either s' or v' is true at w' on m' . Then (11) $\lceil \Box \sim s \rceil$ and $\lceil \Box \sim v \rceil$ are not both true at w on m' . By (9), (11), and Definition 3 (taking q to be s' and r to be v' , and noting that both $\lceil \Box(s' \rightarrow s) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$ are true at w on m'), t favors s' over v' at w on m' . This contradicts (6), and the reductio is complete.

(3A) *Mixed-premise declarative arguments: sufficient condition for strong and for weak semantic c-entailment.* Suppose that (1) $\lceil t \ \& \ \Diamond \sim v \rceil$ semantically c -entails t' . Take any c -interpretation m , any world w of m , and any declarative sentence p . Suppose that (2) p guarantees t at w on m (i.e., both p and $\lceil \Box(p \rightarrow t) \rceil$ are true at w on m) and (3) p weakly supports i at w on m (if p strongly supports i at w on m , then (3) also holds by Definition 3, and the proof proceeds as below). By (3) and Definition 3, there is an imperative sentence i' such that (4) p strongly supports i' at w on m and (5) both $\lceil \Box(s' \rightarrow s) \rceil$ and $\lceil \Box(a' \leftrightarrow a) \rceil$ are true at w on m . By (4) and Definition 3, (6) $\lceil \Box v \rceil$ is not true at w on m . Suppose, for reductio, that $\lceil \Box v \rceil$ is true at w on m . Then both $\lceil \Box \sim s \rceil$ and $\lceil \Box \sim a \rceil$ are true at w on m , so (by (5)) both $\lceil \Box \sim s \rceil$ and $\lceil \Box \sim a \rceil$ are true at w on m , and then $\lceil \Box v \rceil$ is true at w on m . This contradicts (6), and the reductio is complete: $\lceil \Box v \rceil$ is not true at w on m , so $\lceil \Diamond \sim v \rceil$ is true at w on m . Then, given that c entails ε , $\lceil \Box \Diamond \sim v \rceil$ is true at w on m , and thus so is $\lceil \Box(p \rightarrow \Diamond \sim v) \rceil$. Then, by (2), both p and $\lceil \Box(p \rightarrow (t \ \& \ \Diamond \sim v)) \rceil$ are true at w on m ; i.e., p guarantees $\lceil t \ \& \ \Diamond \sim v \rceil$ at w on m . Then, by (1) and Definition 4c, p guarantees t' at w on m . Then, by Definition 4c, t and i together weakly semantically c -entail t' .

(3B) *Mixed-premise declarative arguments: necessary condition for strong and for weak semantic c-entailment.* Suppose, for reductio, that (1) t and i together strongly semantically c -entail t' (if they weakly semantically c -entail t' , then (1) also holds by Definition 4c,⁴² and the proof proceeds as below) but (2) $\lceil t \ \& \ \Diamond \sim v \rceil$ does not semantically c -entail t' . By (2) and the first part of the lemma, there is a c -interpretation m and a world w of m such that (3) $\lceil t \ \& \ \Diamond \sim v \rceil$ is true at w on m but (4) t' is not true at w on m . Consider an interpretation m' that (5) has the same first three coordinates as m but (6) has as its fourth coordinate a function that assigns to w the set of ordered triples $\langle p, q, r \rangle$ such that (a) p is any declarative sentence logically equivalent to t , (b) both $\lceil \Box(q \rightarrow s) \rceil$ and $\lceil \Box(r \rightarrow v) \rceil$ are true at w on m , and (c) $\lceil \Box \sim q \rceil$ and $\lceil \Box \sim r \rceil$ are not both true at w on m .⁴³ If one replaces ' m ' with ' m' ' everywhere in the formulations of (3), (4), and (6), then one obtains formulations of claims—call those claims (3'), (4'), and (6') respectively—that (by (5)) also hold. By (3'), both (7) t is true at w on m' and (8) $\lceil \Box v \rceil$ is not true at w on m' . Then t both guarantees t at w on m' (by (7)) and strongly supports i at w on m' (by (7), (8), (6'), and Definition 3), and thus (by (1) and Definition 4c) t guarantees t' at w on m' . Then, by (7), and given that c entails ρ , t' is true at w on m' . This contradicts (4'), and the reductio is complete.

(4aA) *Mixed-premise imperative arguments: sufficient condition for strong semantic c-entailment.* Suppose that (1) $\lceil t \ \& \ \Diamond \sim v \rceil$ semantically c -entails both $\lceil \Box(s' \rightarrow s) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$. Take any c -interpretation m , any world w of m , and any declarative sentence p . Suppose that (2) p guarantees t at w on m and (3) p strongly supports i at w on m , so that (4) p is true at w on m . By (2), and given that c entails ρ , t is true at w on m . By (3) and Definition 3, (5) $\lceil \Box v \rceil$ is not true at w on m ; then $\lceil \Diamond \sim v \rceil$ is true at w on m , and thus so is $\lceil t \ \& \ \Diamond \sim v \rceil$. Then, by (1) and the first part of the lemma, (6) both $\lceil \Box(s' \rightarrow s) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$ are true at w on m . By (5) and (6): (7) $\lceil \Box v \rceil$ is not true at w on m . Moreover, (8) for any declarative sentences q and r such that $\lceil \Box \sim q \rceil$ and $\lceil \Box \sim r \rceil$ are not both true at w on m , if both $\lceil \Box(q \rightarrow s) \rceil$ and $\lceil \Box(r \rightarrow v) \rceil$ are true at w on m ,

⁴² Indeed: if every declarative sentence that both guarantees t and weakly supports i at w on m also guarantees t' at w on m , then every declarative sentence that both guarantees t and strongly supports i at w on m also guarantees t' at w on m .

⁴³ This set of ordered triples satisfies the asymmetry condition (§2.2): if one supposes for reductio that $\langle p, q, r \rangle$ and $\langle p, r, q \rangle$ are both in the set, then one gets that $\lceil \Box \sim q \rceil$ and $\lceil \Box \sim r \rceil$ are both true at w on m (contradicting (c)): $\lceil \Box \sim q \rceil$ is true at w on m because both $\lceil \Box(q \rightarrow s) \rceil$ and $\lceil \Box(q \rightarrow v) \rceil$ are true at w on m (and $\lceil s \ \& \ v \rceil$ is a contradiction), and similarly for $\lceil \Box \sim r \rceil$. The intensionality condition (§2.2) is also satisfied.

then (by (6)) both $\lceil \Box(q \rightarrow s) \rceil$ and $\lceil \Box(r \rightarrow v) \rceil$ are true at w on m , and then (by (3) and Definition 3), p favors q over r at w on m . By (4), (7), (8), and Definition 3, p strongly supports i' at w on m . Then, by Definition 4c, t and i together strongly semantically c -entail i' .

(4aB) *Mixed-premise imperative arguments: necessary condition for strong semantic c -entailment.* Suppose, for reductio, that (1) t and i together strongly semantically c -entail i' but (2) $\lceil t \ \& \ \Diamond \sim v \rceil$ does not semantically c -entail both $\lceil \Box(s' \rightarrow s) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$. By (2) and the first part of the lemma, there is a c -interpretation m and a world w of m such that (3) $\lceil t \ \& \ \Diamond \sim v \rceil$ is true at w on m but (4) $\lceil \Box(s' \rightarrow s) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$ are not both true at w on m . Consider an interpretation m' that satisfies (5) and (6) as in part 3B of the proof. Then (by (5)) claims (3'), (4'), and (6'), formulated by replacing ' m ' with ' m' ' everywhere in the formulations of (3), (4), and (6) respectively, also hold. By (3'), both (7) t is true at w on m' and (8) $\lceil \Box v \rceil$ is not true at w on m' . Then t both guarantees t at w on m' (by (7)) and strongly supports i at w on m' (by (7), (8), (6'), and Definition 3), and thus (by (1) and Definition 4c): (9) t strongly supports i' at w on m' . Let q be $\lceil s' \ \& \ \sim s \rceil$ and r be $\lceil v' \ \& \ \sim v \rceil$. By (4'), $\lceil \Box \sim q \rceil$ and $\lceil \Box \sim r \rceil$ are not both true at w on m' . Moreover, both $\lceil \Box(q \rightarrow s) \rceil$ and $\lceil \Box(r \rightarrow v) \rceil$ are true at w on m' . Then, by (9) and Definition 3, t favors q over r at w on m' . Then, by (6'), both $\lceil \Box(q \rightarrow s) \rceil$ (i.e., $\lceil \Box((s' \ \& \ \sim s) \rightarrow s) \rceil$; equivalently, $\lceil \Box(s' \rightarrow s) \rceil$) and $\lceil \Box(r \rightarrow v) \rceil$ (i.e., $\lceil \Box((v' \ \& \ \sim v) \rightarrow v) \rceil$; equivalently, $\lceil \Box(v' \rightarrow v) \rceil$) are true at w on m' . This contradicts (4'), and the reductio is complete.

(4bA) *Mixed-premise imperative arguments: sufficient condition for weak semantic c -entailment.* Suppose that (1) t semantically c -entails both $\lceil \Box(a \rightarrow a) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$. Take any c -interpretation m , any world w of m , and any declarative sentence p . Suppose that (2) p guarantees t at w on m and (3) p weakly supports i at w on m . By (3) and Definition 3, there is an imperative sentence i^* such that (4) p strongly supports i^* at w on m and (5) both $\lceil \Box(s^* \rightarrow s) \rceil$ and $\lceil \Box(a^* \leftrightarrow a) \rceil$ are true at w on m for some satisfaction sentence s^* and some avoidance sentence a^* of i^* . Let k be $\lceil \sim a' \rightarrow !(s^* \ \& \ s) \rceil$, and let s_k , v_k , and a_k be respectively the satisfaction sentence $\lceil \sim a' \ \& \ (s^* \ \& \ s) \rceil$, the violation sentence $\lceil \sim a' \ \& \ \sim(s^* \ \& \ s) \rceil$, and the avoidance sentence a' of k . Then (6) both $\lceil \Box(s_k \rightarrow s) \rceil$ and $\lceil \Box(a_k \leftrightarrow a) \rceil$ are true at w on m . Moreover, as one can show by using (1), (5), and the first part of the lemma (cf. Vranas 2011: 436 n. 68), $\lceil t \ \& \ \Diamond \sim v^* \rceil$ semantically c -entails both $\lceil \Box(s_k \rightarrow s^*) \rceil$ and $\lceil \Box(v_k \rightarrow v^*) \rceil$, where v^* is a violation sentence of i^* . Then, by part 4a of the lemma, t and i^* together strongly semantically c -entail k . Then, by (2), (4), and Definition 4c, p strongly supports k at w on m . Then, by (6) and Definition 3, p weakly supports i' at w on m . Then, by Definition 4c, t and i together weakly semantically c -entail i' .

(4bB) *Mixed-premise imperative arguments: necessary condition for weak semantic c -entailment.* Suppose, for reductio, that (1) t and i together weakly semantically c -entail i' but (2) t does not semantically c -entail both $\lceil \Box(a \rightarrow a) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$. By (2) and the first part of the lemma, there is a c -interpretation m and a world w of m such that (3) t is true at w on m but (4) $\lceil \Box(a \rightarrow a) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$ are not both true at w on m . Consider an interpretation m' that satisfies (5) and (6) as in part 3B of the proof. Then (by (5)) claims (3'), (4'), and (6'), formulated by replacing ' m ' with ' m' ' everywhere in the formulations of (3), (4), and (6) respectively, also hold. By (4'): (7) $\lceil \Box v \rceil$ is not true at w on m' (because, if it is, then so are both $\lceil \Box(v' \rightarrow v) \rceil$ and $\lceil \Box \sim a \rceil$, and thus so is also $\lceil \Box(a \rightarrow a) \rceil$). Then t both guarantees t at w on m' (by (3')) and (strongly and thus) weakly supports i at w on m' (by (3'), (7), (6'), and Definition 3), and thus (by (1) and Definition 4c) t weakly supports i' at w on m' . Then, by Definition 3, there is an imperative sentence i^* such that (8) t strongly supports i^* at w on m' and (9) both $\lceil \Box(s^* \rightarrow s) \rceil$ and

$\lceil \Box(a^* \leftrightarrow a) \rceil$ are true at w on m' for some satisfaction sentence s^* and some avoidance sentence a^* of i^* . Then $\lceil \Box((\sim s' \& \sim a) \rightarrow (\sim s^* \& \sim a^*)) \rceil$ is true at w on m' , and thus (10) so is $\lceil \Box(v' \rightarrow v^*) \rceil$. By (4') and (9), $\lceil \Box \sim s^* \rceil$ and $\lceil \Box \sim v^* \rceil$ are not both true at w on m' (because, if they are, then so are $\lceil \Box a^* \rceil$ and (by (9)) $\lceil \Box a \rceil$, and then so are both $\lceil \Box(a \rightarrow a) \rceil$ and $\lceil \Box \sim v \rceil$, and thus so is also $\lceil \Box(v' \rightarrow v) \rceil$). Then, by (8) and Definition 3, t favors s^* over v^* at w on m' . Then, by (6'), both $\lceil \Box(s^* \rightarrow s) \rceil$ and $\lceil \Box(v^* \rightarrow v) \rceil$ are true at w on m' . Then, by (10): (11) $\lceil \Box(v' \rightarrow v) \rceil$ is true at w on m' . Moreover, $\lceil \Box((\sim s \& \sim v) \rightarrow (\sim s^* \& \sim v^*)) \rceil$ is true at w on m' , and thus so is $\lceil \Box(a \rightarrow a^*) \rceil$. Then, by (9): (12) $\lceil \Box(a \rightarrow a) \rceil$ is true at w on m' . But (11) and (12) together contradict (4'), and the reductio is complete.

(5) *Cross-species declarative arguments.* For any c -interpretation m , any world w of m , and any declarative sentence p , p strongly supports i at w on m if and only if p both guarantees $\lceil t \vee \sim t \rceil$ and strongly supports i at w on m : the “if” part is immediate, and the “only if” part holds because, if p strongly supports i at w on m , then p is true at w on m and thus p guarantees $\lceil t \vee \sim t \rceil$ at w on m . Then, by Definition 4c, i strongly semantically c -entails t' exactly if $\lceil t \vee \sim t \rceil$ and i together strongly semantically c -entail t' , and thus (by the third part of the lemma) exactly if $\lceil (t \vee \sim t) \& \Diamond \sim v \rceil$ semantically c -entails t' , and thus (by the first part of the lemma) exactly if $\lceil \Diamond \sim v \rceil$ semantically c -entails t' . Similarly for weak semantic c -entailment.

(6) *Pure imperative arguments.* Reasoning as in the previous paragraph, i strongly semantically c -entails i' exactly if $\lceil t \vee \sim t \rceil$ and i together strongly semantically c -entail i' , and thus (by part 4a of the lemma) exactly if $\lceil (t \vee \sim t) \& \Diamond \sim v \rceil$ semantically c -entails both $\lceil \Box(s' \rightarrow s) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$, and thus exactly if $\lceil \Diamond \sim v \rceil$ semantically c -entails both $\lceil \Box(s' \rightarrow s) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$. Similarly, i weakly semantically c -entails i' exactly if $\lceil t \vee \sim t \rceil$ semantically c -entails both $\lceil \Box(a \rightarrow a) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$, and thus exactly if: (1) for any c -interpretation m and any world w of m , both $\lceil \Box(a \rightarrow a) \rceil$ and $\lceil \Box(v' \rightarrow v) \rceil$ are true at w on m . Given that c entails ρ , (1) holds exactly if: (2) for any c -interpretation m and any world w of m , both $\lceil a \rightarrow a \rceil$ and $\lceil v' \rightarrow v \rceil$ are true at w on m . And (2) holds exactly if both a semantically c -entails a' and v' semantically c -entails v .

A.2.2. Proof of the Soundness and Completeness Theorem

Take any non-empty finite set Γ of sentences, any sentence ϕ , and any constraint c that entails ρ (and also entails ε if both ϕ is a declarative sentence and Γ contains an imperative sentence).

Proof of soundness. For brevity, I prove together the two claims that, if ϕ is (1) strongly c -derivable or (2) c -derivable from Γ , then, respectively, Γ (1) strongly or (2) weakly semantically c -entails ϕ . The proof is by induction on the number of lines of a strong c -derivation or c -derivation. For the base step, suppose there is a one-line (case 1) strong c -derivation or (case 2) c -derivation of ϕ from Γ . In case 1, ϕ is a conjunction of declarative or of all imperative members of Γ and thus (as one can show by using Definition 4c) Γ strongly semantically c -entails ϕ . In case 2, ϕ is (a member or) a conjunction of declarative or of imperative members of Γ ; so, if ϕ is not a conjunction of declarative or of *all* imperative members of Γ (if it is, the proof proceeds as in case 1), there is a conjunction k of the remaining imperative members of Γ , and $\lceil \phi \& k \rceil$ is a conjunction of all imperative members of Γ (see note 8). Then Γ weakly semantically c -entails ϕ because, by Definition 4c, Γ weakly semantically c -entails $\lceil \phi \& k \rceil$, and by part 6b of the Equivalence Lemma, $\lceil \phi \& k \rceil$ weakly semantically c -entails ϕ (see the end of the base step in §A.1.4).

For the inductive step, take any non-zero natural number n and suppose (*induction hypothesis*) that, if there is a (case 1) strong c -derivation or (case 2) c -derivation with at most n lines of ϕ

from Γ , then Γ (case 1) strongly or (case 2) weakly semantically c -entails φ . To complete the proof, take any (case 1) strong c -derivation or (case 2) c -derivation with at most $n + 1$ lines of φ from Γ . Suppose that φ is not a conjunction of declarative or of (case 1) all imperative or (case 2) imperative members of Γ (if it is, the proof proceeds as in the base step). Then φ can be obtained from—a set Δ of—previous lines either by using natural deduction for system Kc of propositional modal logic or by applying once a replacement rule from Table 1, an applicable inference rule from Table 4 (except for MWC in case 1), or: in case 1, c -SA or ECQ; in case 2, c -SA, c -WC, or any inference rule from SPIL except for SA and WC. For any n' -th line k in Δ , $n' \leq n$, so by the induction hypothesis and the fact that the sequence of the first n' lines of the (case 1) strong c -derivation or (case 2) c -derivation of φ from Γ is a (case 1) strong c -derivation or (case 2) c -derivation of k from Γ , it follows that (1) Γ (case 1) strongly or (case 2) weakly semantically c -entails k (for any member k of Δ). By using the Replacement Lemma and the Equivalence Lemma, one can show that all replacement and inference rules that may be applied in (case 1) strong c -derivations or (case 2) c -derivations always correspond to (case 1) strongly or (case 2) weakly semantically c -valid arguments;⁴⁴ moreover, so does using natural deduction for system Kc of propositional modal logic (since it is assumed that a *sound* natural deduction system is used). It follows that (2) Δ (case 1) strongly or (case 2) weakly semantically c -entails φ . By (1) and Definition 4c: (3) for any c -interpretation m and any world w of m , if a declarative sentence p both guarantees at w on m every conjunction of all declarative members of Γ and (case 1) strongly or (case 2) weakly supports at w on m every conjunction of all imperative members of Γ , then p both guarantees at w on m every declarative member (and thus every conjunction of all declarative members) of Δ and (case 1) strongly or (case 2) weakly supports at w on m every imperative member (and thus, as I prove in a note, every conjunction of all imperative members) of Δ .⁴⁵ By (2) and Definition 4c: (4) for any c -interpretation m and any world w of m , if a declarative sentence p both guarantees at w on m every conjunction of all declarative members of

⁴⁴ Concerning replacement rules, they are all based on logical equivalences (as one can show by using C1*-C15*; cf. note 34), so their application preserves logical equivalence (by the Replacement Lemma) and thus also both strong and weak semantic c -entailment (by parts 1 and 6 of the Equivalence Lemma). Concerning inference rules, take for example MSA. If s is a satisfaction sentence of i and v is a violation sentence of i , then (by C12*) $\lceil p \ \& \ s \rceil$ is a satisfaction sentence of $\lceil p \rightarrow i \rceil$ and $\lceil p \ \& \ v \rceil$ is a violation sentence of $\lceil p \rightarrow i \rceil$ (and similarly for $\lceil p' \rightarrow i \rceil$). Then, by part 4a of the Equivalence Lemma, for any c that entails ρ , the argument from $\lceil \Box(p' \rightarrow p) \rceil$ and $\lceil p \rightarrow i \rceil$ to $\lceil p' \rightarrow i \rceil$ is strongly semantically c -valid exactly if: (1) $\lceil \Box(p' \rightarrow p) \rceil \ \& \ \Diamond \sim(p \ \& \ v) \rceil$ semantically c -entails both $\lceil \Box((p' \ \& \ s) \rightarrow (p \ \& \ s)) \rceil$ and $\lceil \Box((p' \ \& \ v)) \rightarrow (p \ \& \ v) \rceil$. By propositional modal logic (system $K\rho$), (1) holds. Similarly for the remaining inference rules. Note, concerning the second part of XECQ, that $\lceil \lceil p \ \& \ \sim p \rceil \rceil$ strongly semantically c -entails q for any c because, by Definition 3, no declarative sentence strongly supports $\lceil \lceil p \ \& \ \sim p \rceil \rceil$ at any world on any interpretation. Note also, concerning PNV, that part 3A of the proof of the Equivalence Lemma uses the assumption that c entails ε but not the assumption that c entails ρ (the latter assumption is used in part 3B of the proof).

⁴⁵ In case 1, Δ has at most one imperative member: no rule that may be applied in strong c -derivations has more than one imperative premise. For case 2, suppose for example that Δ has two imperative members, k' and k'' . Then Γ weakly semantically c -entails both k' and k'' . There are three cases to consider. Case 2a: Γ has both declarative and imperative members. Take any conjunctions t and i of all declarative and of all imperative members of Γ respectively, and any avoidance sentences a , a' , and a'' and violation sentences v , v' , and v'' of i , k' , and k'' respectively. By part 4b of the Equivalence Lemma, t semantically c -entails all of $\lceil \Box(a \rightarrow a') \rceil$, $\lceil \Box(a \rightarrow a'') \rceil$, $\lceil \Box(v' \rightarrow v) \rceil$, and $\lceil \Box(v'' \rightarrow v) \rceil$, and thus also both $\lceil \Box(a \rightarrow (a' \ \& \ a'')) \rceil$ and $\lceil \Box((v' \ \vee \ v'') \rightarrow v) \rceil$. By C10*, $\lceil a' \ \& \ a'' \rceil$ is an avoidance sentence and $\lceil v' \ \vee \ v'' \rceil$ is a violation sentence of $\lceil k' \ \& \ k'' \rceil$. Then, by part 4b of the Equivalence Lemma, t and i together weakly semantically c -entail $\lceil k' \ \& \ k'' \rceil$, and thus so does Γ . Case 2b: Γ has only declarative members. Then, by the second part of the Equivalence Lemma, t semantically c -entails both $\lceil \Box a \rceil$ and $\lceil \Box a' \rceil$, and thus also $\lceil \Box(a' \ \& \ a'') \rceil$, so t weakly semantically c -entails $\lceil k' \ \& \ k'' \rceil$, and thus so does Γ . Case 2c: Γ has only imperative premises. Then reason as in note 36.

Δ and (case 1) strongly or (case 2) weakly supports at w on m every conjunction of all imperative members of Δ , then p (case 1) strongly or (case 2) weakly sustains ϕ at w on m . By (3), (4), and Definition 4c, Γ (case 1) strongly or (case 2) weakly semantically c -entails ϕ .

Proof of completeness. If Γ has a declarative member, let t be a conjunction of all declarative members of Γ . If Γ has an imperative member, let i be a conjunction of all imperative members of Γ , and let p and q be declarative sentences such that i and $\lceil p \rightarrow !q \rceil$ are (1) replacement interderivable (such declarative sentences exist by the Canonical Form Lemma) and thus also (2) logically equivalent (by the Replacement Lemma and the fact that all replacement rules are based on logical equivalences). Similarly, if ϕ is an imperative sentence, let p' and q' be declarative sentences such that ϕ and $\lceil p' \rightarrow !q' \rceil$ are (3) replacement interderivable and (4) logically equivalent. There are six cases to consider, corresponding to the six parts of the Equivalence Lemma.

Case 1: Pure declarative arguments. Completeness holds because it is assumed that a *complete* natural deduction system for propositional modal logic is used. (In what follows, I refer to this assumption as “PML-completeness”.)

Case 2: Cross-species imperative arguments. Suppose that Γ strongly (equivalently: weakly) semantically c -entails ϕ . Then, by (4) and the second part of the Equivalence Lemma, (5) t semantically (and thus also syntactically, by PML-completeness) c -entails $\lceil \Box \sim p \rceil$. To conclude: there is a (strong) c -derivation from Γ of t , and thus also, by (5), of $\lceil \Box \sim p \rceil$, and thus also, by MEI, of $\lceil p' \rightarrow !q' \rceil$, and thus finally, by (3), of ϕ .

Case 3: Mixed-premise declarative arguments. Suppose that Γ strongly (equivalently: weakly) semantically c -entails ϕ . Then, by (2) and the third part of the Equivalence Lemma, (6) $\lceil t \ \& \ \Diamond(p \rightarrow q) \rceil$ semantically (and thus also syntactically, by PML-completeness) c -entails ϕ (recall that $\lceil p \rightarrow q \rceil$ is a non-violation sentence of $\lceil p \rightarrow !q \rceil$). To conclude: there is a (strong) c -derivation from Γ of i , and thus also, by (1), of $\lceil p \rightarrow !q \rceil$, and thus also, by PNV, of $\lceil \Diamond(p \rightarrow q) \rceil$, and thus also (by adding t as a line right after $\lceil \Diamond(p \rightarrow q) \rceil$ and applying, e.g., CI) of $\lceil t \ \& \ \Diamond(p \rightarrow q) \rceil$, and thus finally, by (6), of ϕ .

Case 4: Mixed-premise imperative arguments. Suppose first that Γ *strongly* semantically c -entails ϕ . Then, by (2), (4), and part 4a of the Equivalence Lemma: (7) $\lceil t \ \& \ \Diamond(p \rightarrow q) \rceil$ semantically (and thus also syntactically, by PML-completeness) c -entails both $\lceil \Box((p' \ \& \ q') \rightarrow (p \ \& \ q)) \rceil$ and $\lceil \Box((p' \ \& \ \sim q') \rightarrow (p \ \& \ \sim q)) \rceil$. Moreover, as in case 3, (8) there is a strong c -derivation from Γ of $\lceil t \ \& \ \Diamond(p \rightarrow q) \rceil$. By (1), (7), and (8), there is a strong c -derivation from Γ which includes the first three lines of the following sequence (which is a strong c -derivation from its first three lines):

- | | |
|--|--|
| 1. $p \rightarrow !q$ | |
| 2. $\Box((p' \ \& \ q') \rightarrow (p \ \& \ q))$ | |
| 3. $\Box((p' \ \& \ \sim q') \rightarrow (p \ \& \ \sim q))$ | |
| 4. $\Box(p' \rightarrow p)$ | 2, 3 Propositional Modal Logic (system Kc) |
| 5. $p' \rightarrow !q$ | 1, 4 Modally Strengthening the Antecedent |
| 6. $p' \rightarrow !(p' \ \& \ q)$ | 5 Absorption |
| 7. $\Box((p' \ \& \ q) \leftrightarrow (p' \ \& \ q'))$ | 2, 3 Propositional Modal Logic (system Kc) |
| 8. $p' \rightarrow !(p' \ \& \ q')$ | 6, 7 Modally Equivalent Consequent |
| 9. $p' \rightarrow !q'$ | 8 Absorption |

There is then a strong c -derivation from Γ of $\lceil p' \rightarrow !q' \rceil$, and thus also, by (3), of ϕ .

Suppose next that Γ *weakly* semantically c -entails ϕ . Then, by (2), (4), and part 4b of the Equivalence Lemma: (9) t semantically (and thus also syntactically, by PML-completeness) c -entails both $\lceil \Box(\sim p \rightarrow \sim p) \rceil$ and $\lceil \Box((p' \& \sim q) \rightarrow (p \& \sim q)) \rceil$. Moreover, (10) there is a c -derivation from Γ of t . By (1), (9), and (10), there is a c -derivation from Γ which includes the first three lines of the following sequence (which is a c -derivation from its first three lines):

- | | |
|---|---|
| 1. $p \rightarrow !q$ | |
| 2. $\Box(\sim p \rightarrow \sim p)$ | |
| 3. $\Box((p' \& \sim q) \rightarrow (p \& \sim q))$ | |
| 4. $\Box(p' \rightarrow p)$ | 2 Propositional Modal Logic (system Kc) |
| 5. $p' \rightarrow !q$ | 1, 4 Modally Strengthening the Antecedent |
| 6. $p' \rightarrow !(p' \& q)$ | 5 Absorption |
| 7. $\Box((p' \& q) \rightarrow (p' \& q))$ | 3 Propositional Modal Logic (system Kc) |
| 8. $p' \rightarrow !(p' \& q)$ | 6, 7 Modally Weakening the Consequent |
| 9. $p' \rightarrow !q'$ | 8 Absorption |

There is then a c -derivation from Γ of $\lceil p' \rightarrow !q \rceil$, and thus also, by (3), of ϕ .

Case 5: Cross-species declarative arguments. Suppose that Γ strongly (equivalently: weakly) semantically c -entails ϕ . Then, by (2) and the fifth part of the Equivalence Lemma, (11) $\lceil \Diamond(p \rightarrow q) \rceil$ semantically (and thus also syntactically, by PML-completeness) c -entails ϕ . To conclude: there is a (strong) c -derivation from Γ of i , and thus also, by (1), of $\lceil p \rightarrow !q \rceil$, and thus also, by PNV, of $\lceil \Diamond(p \rightarrow q) \rceil$, and thus finally, by (11), of ϕ .

Case 6: Pure imperative arguments. Suppose that Γ *strongly* semantically c -entails ϕ . Then $\lceil p \vee \sim p \rceil$ and i together strongly semantically c -entail ϕ (see part 6 of the proof of the Equivalence Lemma). Then, by case 4 above, (12) there is a strong c -derivation from $\lceil p \vee \sim p \rceil$ and i of ϕ . Moreover, (13) there is a strong c -derivation from Γ which includes both i and $\lceil p \vee \sim p \rceil$ (the latter can be obtained by using natural deduction for system Kc of propositional modal logic). By (12) and (13), there is a strong c -derivation from Γ of ϕ . Similarly if Γ *weakly* semantically c -entails ϕ .

A.3. Quantified pure imperative logic (QPIL)

My main goal here is to prove the following theorem:

SOUNDNESS AND COMPLETENESS THEOREM (FOR QPIL).

- (1) Sentences ϕ and ψ are logically equivalent exactly if they are replacement interderivable.
- (2) A pure imperative argument is strongly semantically valid exactly if it is strongly syntactically valid, and is weakly semantically valid exactly if it is weakly syntactically valid.

My proof of the theorem is very similar to the proof of the corresponding theorem for SPIL in §A.1, and uses the Canonical Form Lemma (for QPIL), the Replacement Lemma (for QPIL), and the Equivalence Lemma (for QPIL). These three lemmata have formulations identical to the formulations of the corresponding lemmata for SPIL in §A.1 (except that, for the Replacement Lemma, “subsentence” is replaced with “subformula”, and ψ and ψ' are formulas that need not be sentences), although here one quantifies over sentences and interpretations of the language of QPIL instead of SPIL.

The proofs of the Canonical Form Lemma and of the Replacement Lemma are the same as the proofs in §A.1.1 and §A.1.2 respectively, except that (a) “connectives”, “sentence letter”, and

“subsentence” are replaced with “connectives or quantifiers”, “atomic sentence”, and “subformula” respectively (and, for the Replacement Lemma, ψ and ψ' are formulas that need not be sentences), and (b) in the inductive step, there is one more case to consider (case 8): for some variable u and some imperative formula j , i is either $\ulcorner \forall u j \urcorner$ or $\ulcorner \exists u j \urcorner$. Concerning the Canonical Form Lemma: in case 8, by the induction hypothesis, j is replacement interderivable with $\ulcorner p \rightarrow !q \urcorner$, and then, by IQ, i is replacement interderivable either with $\ulcorner \exists u p \rightarrow !\forall u(p \rightarrow q) \urcorner$ or with $\ulcorner \exists u p \rightarrow !\exists u(p \& q) \urcorner$.⁴⁶ Concerning the Replacement Lemma: in case 8, ψ is a subformula of j , and i' is either $\ulcorner \forall u j \urcorner$ or $\ulcorner \exists u j \urcorner$, where j' is $j(\psi/\psi')$ and thus, by the induction hypothesis, is logically equivalent to j . It follows that, for any interpretation m and any member o of the domain of m , (1) o satisfies j on m exactly if o satisfies j' on m (and similarly for violating and avoiding): if o satisfies j on m , then, for any constant h that occurs neither in j nor in j' , $j[u/h]$ is satisfied on $m[h/o]$, so $j'[u/h]$ is also satisfied on $m[h/o]$ (since j and j' are logically equivalent), and thus o satisfies j' on m (and, similarly, vice versa). By (1), C22, and C23, $\ulcorner \forall u j \urcorner$ and $\ulcorner \forall u j' \urcorner$ are logically equivalent and so are $\ulcorner \exists u j \urcorner$ and $\ulcorner \exists u j' \urcorner$, and thus so are also i and i' .

The proof of the Equivalence Lemma is the same as the proof in §A.1.3, and the proof of completeness is the same as the corresponding proof in §A.1.4.⁴⁷ The proof of soundness concerning logical equivalence and strong semantic validity is the same as the corresponding proof in §A.1.4, except that one must also show that all replacement rules in Table 5 are based on logical equivalences. One can show this by using C1-C15 and C18-C23 (§4.2); I give an example in a note.⁴⁸ It only remains then to prove soundness concerning weak semantic validity. Here the

⁴⁶ By replacing in this proof “sentence” and “sentences” with “formula” and “formulas” respectively, one obtains a proof of the following generalization of the Canonical Form Lemma: for any imperative *formula* i , there are declarative *formulas* p and q such that i and $\ulcorner p \rightarrow !q \urcorner$ are replacement interderivable (redefining replacement derivations so as to allow their lines to be any formulas, not just sentences). One can also prove (although with more significant modifications to the proof of the Replacement Lemma) the following generalization of the Replacement Lemma: for any *formulas* ϕ , ψ , and ψ' such that ψ is a subformula of ϕ and ψ is logically equivalent to ψ' , ϕ is logically equivalent to any formula that results from replacing in ϕ at least one occurrence of ψ with ψ' . Finally, by using these two generalizations, one can prove the following result: for any imperative formula i , there are declarative formulas p and q such that i and $\ulcorner p \rightarrow !q \urcorner$ are logically equivalent.

⁴⁷ Just as in §A.1.4, I assume that any logically equivalent *declarative* sentences are replacement interderivable, since this corresponds to a result from classical logic (cf. note 37). Here is a sketch of a proof, due to Jeremy Avigad (personal communication). Take any logically equivalent declarative sentences p and q . By TC, (1) p and $\ulcorner (p \vee \sim p) \& p \urcorner$ are replacement interderivable, and (2) so are q and $\ulcorner (p \vee \sim p) \& q \urcorner$. Assume (this is proved below) that (3) any declarative tautologies are replacement interderivable. Then $\ulcorner p \leftrightarrow q \urcorner$ and $\ulcorner p \vee \sim p \urcorner$ are replacement interderivable, and thus so are (by (1)) p and $\ulcorner (p \leftrightarrow q) \& p \urcorner$ and (by (2)) q and $\ulcorner (p \leftrightarrow q) \& q \urcorner$. But $\ulcorner (p \leftrightarrow q) \& p \urcorner$ and $\ulcorner (p \leftrightarrow q) \& q \urcorner$ are replacement interderivable (as one can show by using ME, CO, DI, AS, IP, CC, and CD), and thus so are p and q . To prove (3), use a result from proof theory, namely that the “one-sided” Gentzen-Schütte sequent calculus GS3 (Troelstra & Schwichtenberg 2000: 86) is complete, so (4) there is a proof of any declarative tautology (in “negation normal form”) from axioms of GS3 using the rules of GS3. The axioms of GS3 amount to disjunctions of the form $\ulcorner (p \vee \sim p) \vee q \urcorner$, so (5) any universal closures of conjunctions of axioms of GS3 are replacement interderivable (by TD, IP, and VQ). (One can add to GS3 axioms for identity; these would be captured by IR and IS.) Moreover, one can show that, for each rule of GS3, any universal closure of a conjunction of all premises of (any instance of) the rule is replacement interderivable with any universal closure of the conclusion of the rule. (Strictly speaking, the premises and the conclusions of instances of the rules of GS3 are finite *sets* of formulas, so replace each set with any disjunction of all its members.) So from (4) it follows that (6) there is a replacement derivation of any declarative tautology from a universal closure of a conjunction of axioms of GS3. Finally, (5) and (6) together entail (3).

⁴⁸ Take the imperative part of QN, and suppose for simplicity that no occurrence of any variable different from both u and u' is free in i . Take any distinct constants h and h' that do not occur in i , and let j be the sentence $i[u/h, u'/h']$. To show that $\ulcorner \sim \forall u j \urcorner$ and $\ulcorner \exists u \sim j \urcorner$ (and thus $\ulcorner \sim \forall u i \urcorner$ and $\ulcorner \exists u \sim i \urcorner$) are logically equivalent, take any interpretation m

proof deviates from the corresponding proof in §A.1.4 (because in QPIL the sequence of the first n lines of a derivation of j from Γ is *not* always a derivation of the n -th line from Γ : a constant introduced by IEI may occur in the n -th line). Given a derivation of j from Γ , call every line that can be obtained only by applying IEI an *assumption*, and say that a line *relies* on an assumption exactly if some constant that occurs in the line occurs first in the assumption (and thus is introduced by IEI). (Every assumption relies on itself.) Say also that a line i *depends* on an assumption i' exactly if there is a finite sequence of lines such that the first member of the sequence is i' , the last member is i , and every member except the first relies on the previous member (i.e., the dependence relation is the *ancestral* of the reliance relation). I will prove that, (1) for any non-zero natural number n that does not exceed the number of lines of the derivation, the n -th line—call it i_n —is weakly semantically entailed by $\Gamma \cup \Theta_n$, where Θ_n is the set of all assumptions on which i_n depends (and thus is empty if i_n is the last line of the derivation, so from (1) it follows that j is weakly semantically entailed by Γ). Let Γ_n be $\Gamma \cup \Theta_n$.

The proof of (1) is by induction on n . For the base step, note that i_1 is (a member or) a conjunction of members of Γ (so $\Theta_1 = \emptyset$), and proceed as in the corresponding proof in §A.1.4. For the inductive step, take any non-zero natural number n less than the number of lines of the derivation, and suppose (*induction hypothesis*) that, for any non-zero natural number n' that does not exceed n , $\Gamma_{n'}$ weakly semantically entails $i_{n'}$. It remains to prove that (2) Γ_{n+1} weakly semantically entails i_{n+1} . This is immediate if i_{n+1} is an assumption (because then i_{n+1} is a member of Θ_{n+1} and thus of Γ_{n+1}) or a conjunction of members of Γ , so suppose i_{n+1} is neither an assumption nor a conjunction of members of Γ . To prove (2), I will prove two claims:

(S1) There is a (maybe empty) set Θ^0 of assumptions earlier than (i.e., previous to) i_{n+1} such that $\Gamma_{n+1} \cap \Theta^0 = \emptyset$ and $\Gamma_{n+1} \cup \Theta^0$ weakly semantically entails i_{n+1} .

(S2) For any non-empty set Θ of assumptions earlier than i_{n+1} such that $\Gamma_{n+1} \cap \Theta = \emptyset$ and $\Gamma_{n+1} \cup \Theta$ weakly semantically entails i_{n+1} , there is a set Θ' of assumptions earlier than i_{n+1} such that $\Gamma_{n+1} \cap \Theta' = \emptyset$, $\Gamma_{n+1} \cup \Theta'$ weakly semantically entails i_{n+1} , and, if $\Theta' \neq \emptyset$, then the latest member of Θ' is earlier (i.e., comes earlier in the derivation) than the latest member of Θ .

Here is why S1 and S2 together entail (2). Suppose, for reductio, that S1 and S2 are true but (2) is false. Then, by S1, $\Theta^0 \neq \emptyset$. Then, by S2 (with $\Theta = \Theta^0$), $\Theta' \neq \emptyset$. Using S2 repeatedly, there is a sequence of more than n non-empty sets of assumptions earlier than i_{n+1} such that, for any set in the sequence except the first, its latest member is earlier than the latest member of the previous set in the sequence. This is impossible, since there are only n lines earlier than i_{n+1} , and the reductio is complete.

and let Δ be its domain. By C9, $\ulcorner \sim \forall u j \urcorner$ is satisfied on m exactly if $\ulcorner \forall u j \urcorner$ is violated on m ; i.e., by C22, exactly if some member of Δ violates j on m ; i.e., by C9, exactly if some member of Δ satisfies $\ulcorner \sim j \urcorner$ on m ; i.e., by C23, exactly if $\ulcorner \exists u \sim j \urcorner$ is satisfied on m (and similarly for violation and avoidance on m). See note 26 for another example. Another way to show that the imperative parts of replacement rules in Table 5 are based on logical equivalences is by using (a) the logical equivalences on which IQ, the rules in Table 1, and the declarative parts of the rules in Table 5 are based, (b) the generalization of the Replacement Lemma given in note 46, and (c) the result given at the end of note 46. For example, take again the imperative part of QN. The formula $\ulcorner \sim \forall u i \urcorner$ is logically equivalent, by (c), to $\ulcorner \sim \forall u (p \rightarrow !q) \urcorner$, and, by IQ (more precisely: by a logical equivalence on which IQ is based), to $\ulcorner \sim (\exists u p \rightarrow !\forall u (p \rightarrow q)) \urcorner$, and, by NC, to $\ulcorner \exists u p \rightarrow \sim !\forall u (p \rightarrow q) \urcorner$, and, by UN, to $\ulcorner \exists u p \rightarrow !\sim \forall u (p \rightarrow q) \urcorner$, and, by the declarative part of QN, to $\ulcorner \exists u p \rightarrow \exists u \sim (p \rightarrow q) \urcorner$, and, by NC, to $\ulcorner \exists u p \rightarrow \exists u (p \& \sim q) \urcorner$, and, by IQ, to $\ulcorner \exists u (p \rightarrow !\sim q) \urcorner$, and, by UN, to $\ulcorner \exists u (p \rightarrow \sim !q) \urcorner$, and, by NC, to $\ulcorner \exists u \sim (p \rightarrow !q) \urcorner$, and finally, by (c), to $\ulcorner \exists u \sim i \urcorner$.

I prove next S1. Let Δ be a set of lines earlier than i_{n+1} from which i_{n+1} can be obtained by applying once a replacement or an inference rule (other than IEI, since i_{n+1} is not an assumption). Let Γ_Δ be the union of all $\Gamma_{n'}$ such that $i_{n'} \in \Delta$ (so $n' \leq n$). By the induction hypothesis, for any $i_{n'}$ in Δ , $\Gamma_{n'}$ weakly semantically entails $i_{n'}$; so (3) Γ_Δ weakly semantically entails every member of Δ , and thus also any conjunction of all members of Δ (cf. note 36). Moreover, all replacement and inference rules that may be applied in derivations always correspond to weakly semantically valid arguments (except for IEI, which is irrelevant here, and IUG, which I address in note 49); so (4) any conjunction of all members of Δ weakly semantically entails i_{n+1} . By (3), (4), and the transitivity of weak semantic entailment, Γ_Δ weakly semantically entails i_{n+1} ,⁴⁹ and thus so does $\Gamma_\Delta \cup \Gamma_{n+1}$. Now let Θ^0 be $\Gamma_\Delta \setminus \Gamma_{n+1}$. Then $\Gamma_{n+1} \cap \Theta^0 = \emptyset$, $\Gamma_{n+1} \cup \Theta^0$ (i.e., $\Gamma_\Delta \cup \Gamma_{n+1}$) weakly semantically entails i_{n+1} , and Θ^0 is a set of assumptions earlier than i_{n+1} (because the members of Θ^0 are the assumptions in Γ_Δ that are not in Γ_{n+1}).

I prove finally S2. Take any non-empty set Θ of assumptions earlier than i_{n+1} such that $\Gamma_{n+1} \cap \Theta = \emptyset$ and $\Gamma_{n+1} \cup \Theta$ weakly semantically entails i_{n+1} . Let $i[u/h]$ be the latest member of Θ , let Θ^* be $\Theta \setminus \{i[u/h]\}$, and let $i_{n'}$ be a line from which $i[u/h]$ can be obtained by applying IEI (so $i_{n'}$ is $\ulcorner \exists u i \urcorner$, and $n' \leq n$). By the induction hypothesis, $\Gamma_{n'}$ weakly semantically entails $i_{n'}$, and thus (5) so does $\Gamma_{n'} \cup (\Gamma_{n+1} \cup \Theta^*)$. Moreover, (6) $\Gamma_{n'} \cup (\Gamma_{n+1} \cup \Theta)$ weakly semantically entails i_{n+1} (since $\Gamma_{n+1} \cup \Theta$ does). By (5) and (6), as I prove in note 50, $\Gamma_{n'} \cup (\Gamma_{n+1} \cup \Theta^*)$ weakly semantically entails i_{n+1} .⁵⁰ Let Θ' be $(\Theta^* \cup \Gamma_{n'}) \setminus \Gamma_{n+1}$. Then $\Gamma_{n+1} \cap \Theta' = \emptyset$, $\Gamma_{n+1} \cup \Theta'$ (i.e., $\Gamma_{n'} \cup (\Gamma_{n+1} \cup \Theta^*)$) weakly semantically entails i_{n+1} , and Θ' is a set of assumptions earlier than i_{n+1} and is such that, if it is non-empty, then its latest member is earlier than the latest member of Θ .

⁴⁹ Concerning IUG, suppose i_{n+1} is of the form $\ulcorner \forall u i \urcorner$ and can be obtained from $\Delta = \{i[u/h]\}$ by applying once IUG. Then $\Gamma_\Delta = \Gamma$: no constant that occurs in $i[u/h]$ is introduced by IEI, so $i[u/h]$ does not depend on any assumption. Let k be any conjunction of all members of Γ . There are declarative formulas p, q, r , and t such that (a) i and k are logically equivalent to $\ulcorner p \rightarrow !q \urcorner$ and $\ulcorner r \rightarrow !t \urcorner$ respectively (by the result given at the end of note 46), and (b) h does not occur in p, q, r , or t (because h does not occur in any member of Γ or in i , and no replacement rule used in the proof of the Canonical Form Lemma introduces new constants). Then $i[u/h]$ is logically equivalent to $\ulcorner p[u/h] \rightarrow !q[u/h] \urcorner$, and $\ulcorner \forall u i \urcorner$ is logically equivalent to $\ulcorner \forall u (p \rightarrow !q) \urcorner$ and thus also to $\ulcorner \exists u p \rightarrow !\forall u (p \rightarrow q) \urcorner$. Since Γ weakly semantically entails $i[u/h]$, $\ulcorner r \rightarrow !t \urcorner$ weakly semantically entails $\ulcorner p[u/h] \rightarrow !q[u/h] \urcorner$. Then, by the Equivalence Lemma, $p[u/h]$ entails r and $\ulcorner p[u/h] \& \sim q[u/h] \urcorner$ entails $\ulcorner r \& \sim t \urcorner$. Then, as in classical first-order logic, $\ulcorner \exists u p \urcorner$ entails r and $\ulcorner \exists u (p \& \sim q) \urcorner$ entails $\ulcorner r \& \sim t \urcorner$. Then, by the Equivalence Lemma, $\ulcorner r \rightarrow !t \urcorner$ weakly semantically entails $\ulcorner \exists u p \rightarrow !\forall u (p \rightarrow q) \urcorner$. So (k and thus) Γ_Δ weakly semantically entails i_{n+1} , and the proof of S1 proceeds as in the text.

⁵⁰ Note first that h does not occur in any member of $\Gamma_{n'} \cup (\Gamma_{n+1} \cup \Theta^*)$: all members of $\Gamma_{n'} \cup (\Gamma \cup \Theta^*)$ are earlier than the latest member of Θ (namely the line in which h is introduced by IEI), and if one supposes for reductio that h occurs in some member of Θ_{n+1} (i.e., $\Gamma_{n+1} \setminus \Gamma$), then that member of Θ_{n+1} depends on $i[u/h]$, and thus so does i_{n+1} (since i_{n+1} depends on every member of Θ_{n+1}), whereas i_{n+1} does not depend on any member of Θ (since $\Gamma_{n+1} \cap \Theta = \emptyset$) and thus does not depend on $i[u/h]$. Let k be any conjunction of all members of $\Gamma_{n'} \cup (\Gamma_{n+1} \cup \Theta^*)$. There are declarative formulas p, q, p', q', r , and t such that (a) i, i_{n+1} , and k are logically equivalent to $\ulcorner p \rightarrow !q \urcorner$, $\ulcorner p' \rightarrow !q' \urcorner$, and $\ulcorner r \rightarrow !t \urcorner$ respectively (by the result given at the end of note 46), and (b) h does not occur in p, q, p', q', r , or t (because no replacement rule used in the proof of the Canonical Form Lemma introduces new constants). Then $i[u/h]$ is logically equivalent to $\ulcorner p[u/h] \rightarrow !q[u/h] \urcorner$, and $\ulcorner \exists u i \urcorner$ is logically equivalent to $\ulcorner \exists u (p \rightarrow !q) \urcorner$ and thus also to $\ulcorner \exists u p \rightarrow !\exists u (p \& q) \urcorner$. Then, by (5), $\ulcorner r \rightarrow !t \urcorner$ weakly semantically entails $\ulcorner \exists u p \rightarrow !\exists u (p \& q) \urcorner$. Then, by the Equivalence Lemma, (7) $\ulcorner \exists u p \urcorner$ entails r and (8) $\ulcorner \exists u p \& \sim \exists u (p \& q) \urcorner$ entails $\ulcorner r \& \sim t \urcorner$. Similarly, by (6), $\ulcorner (r \rightarrow !t) \& (p[u/h] \rightarrow !q[u/h]) \urcorner$ weakly semantically entails $\ulcorner p' \rightarrow !q' \urcorner$. Then, by the Equivalence Lemma and (some of) C2-C15, (9) p' entails $\ulcorner r \vee p[u/h] \urcorner$ and (10) $\ulcorner p' \& \sim q' \urcorner$ entails $\ulcorner (r \& \sim t) \vee (p[u/h] \& \sim q[u/h]) \urcorner$. Then, as in classical first-order logic, p' entails r (by (7) and (9)) and $\ulcorner p' \& \sim q' \urcorner$ entails $\ulcorner r \& \sim t \urcorner$ (by (8) and (10)). Then, by the Equivalence Lemma, $\ulcorner r \rightarrow !t \urcorner$ weakly semantically entails $\ulcorner p' \rightarrow !q' \urcorner$. So (k and thus) $\Gamma_{n'} \cup (\Gamma_{n+1} \cup \Theta^*)$ weakly semantically entails i_{n+1} .

A.4. Quantified modal imperative logic (QMIL)

The Soundness and Completeness Theorem for QMIL has a formulation identical to the formulation of the corresponding theorem for SMIL in §A.2. The Canonical Form Lemma for QMIL and the Replacement Lemma for QMIL have formulations (and proofs) identical to the formulations (and the proofs) of the corresponding lemmata forQPIL in §A.3. The Equivalence Lemma for QMIL has a formulation identical (and a proof very similar) to the formulation (and the proof) of the corresponding lemma for SMIL in §A.2. The proof of soundness concerning strong semantic *c*-validity for QMIL and the proof of completeness for QMIL are very similar to the corresponding proofs for SMIL in § A.2.2. Finally, the proof of soundness concerning weak semantic *c*-validity for QMIL deviates from the corresponding proof for SMIL in ways similar to those in which the proof of soundness concerning weak semantic validity forQPIL deviates from the corresponding proof for SPIL (§A.3).

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