NEW FOUNDATIONS FOR IMPERATIVE LOGIC IV: NATURAL DEDUCTION*

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Abstract. Sentential Pure Imperative Logic (SPIL) deals with arguments from imperative premises to imperative conclusions (i.e., pure imperative arguments) that do not contain quantifiers or modal operators. I introduce a formal language and a natural deduction system for SPIL. I provide the formal language with a semantics, and I prove that the natural deduction system is sound and complete with respect to that semantics.

1. Introduction

In this paper, I present a sound and complete natural deduction system for *Sentential Pure Imperative Logic* (SPIL), which deals with arguments from imperative premises to imperative conclusions but does not include quantifiers or modal operators. I provide an imperative formal language, as well as replacement and inference rules that can be used to derive a conclusion from a set of premises. The replacement and inference rules are intended to represent natural patterns of reasoning, but their justification is not limited to intuitions about naturalness. The justification relies crucially on the result—which I prove—that derivability by those rules corresponds to a semantic definition of argument validity that I have developed at length in previous papers (Vranas 2011, 2016; see also Vranas 2008, 2010, 2013) and that I develop further here by introducing *interpretations* of imperative formal languages. I do not presuppose any familiarity with the previous papers.[1]

2. Syntax

The (imperative formal) language of SPIL has the following symbols: the connectives ‘∼’, ‘&’, ‘∨’, ‘→’, and ‘↔’, the punctuation symbols ‘(’ and ‘)’, the *imperative operator* ‘!’ (“let it be the case that”), and the (infinitely many) sentence letters ‘A’, ‘B’, …, ‘Z’, ‘A'’, ‘B'’, …, ‘Z'’, ‘A''’,

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[1] There is hardly any previous work on this subject. To my knowledge, only two logic textbooks cover symbolization of imperative English sentences and natural deduction for imperative logic: Clarke & Behling 1998 (a descendant of Clarke 1973) and Gensler 2002 (a descendant of Gensler 1990; see also Gensler 1996: 181–6). These textbooks, however, rely on inadequate definitions of validity (see Vranas 2011, 2016). Relying on my definition of validity for arguments with only imperative premises and conclusions (Vranas 2011), Hansen (2014) has provided sound and complete sets of inference rules for a formal language with only one imperative connective. See also Fine 2018: 625–6.

'B''', …. (One could also define languages of SPIL with different sentence letters or with only finitely many sentence letters, but for simplicity I define only a single language of SPIL.) The *declarative sentences* of SPIL can be built up from sentence letters as in classical sentential logic. The *imperative sentences* of SPIL are all and only those finite strings of symbols (understood as ordered n-tuples of symbols) that can be built up from declarative sentences by applying the following formation rules (R1 must be applied at least once):

(R1) If p is a declarative sentence, then $\ulcorner !p \urcorner$ is an imperative sentence.
(R2) If i and j are imperative sentences, then $\ulcorner \sim i \urcorner$, $\ulcorner (i \ \& \ j) \urcorner$, and $\ulcorner (i \vee j) \urcorner$ are also imperative sentences.
(R3) If p is a declarative sentence and i is an imperative sentence, then $\ulcorner (p \rightarrow i) \urcorner$, $\ulcorner (i \rightarrow p) \urcorner$, $\ulcorner (p \leftrightarrow i) \urcorner$, and $\ulcorner (i \leftrightarrow p) \urcorner$ are imperative sentences.

A *sentence* (of SPIL) is either a declarative sentence or an imperative sentence. It follows from these definitions that a sentence is imperative iff it contains at least one occurrence of '!' and is declarative iff it contains no occurrence of '!' (so no sentence is both declarative and imperative). Throughout this paper, I use the following notation: (1) φ and ψ are (declarative or imperative) sentences, (2) p, q, r, p', … are declarative sentences, (3) i, j, k, i', … are imperative sentences, and (4) e is a sentence letter. For simplicity, I usually omit outermost parentheses.

3. Semantics

An *interpretation* of the language of SPIL is an ordered pair $m = \langle \mathbb{S}, \mathbb{F} \rangle$, where $\mathbb{S}$ is a set of sentence letters and $\mathbb{F}$ is a *favoring relation*, namely a three-place relation on declarative sentences that satisfies two conditions. First, the *intensionality condition*: for any p, q, and r, and any p', q', and r' interderivable in classical sentential logic with p, q, and r respectively, $\langle p, q, r \rangle \in \mathbb{F}$ iff $\langle p', q', r' \rangle \in \mathbb{F}$. Second, the *asymmetry condition*: for any p, q, and r, it is not the case that both $\langle p, q, r \rangle \in \mathbb{F}$ and $\langle p, r, q \rangle \in \mathbb{F}$. Informally, a favoring relation corresponds to *comparative reasons* (e.g., reasons for you to marry Hugh *rather than* Hugo), so the asymmetry condition corresponds to the claim that nothing can be a reason both for q rather than r and for r rather than q. The favoring relation is used in §5 to define semantic validity.

On a given interpretation m, a declarative sentence p is *true* ($m \vDash p$) or not ($m \nvDash p$), and an imperative sentence i is *satisfied* ($m \Vdash_s i$) or not ($m \nVdash_s i$); if i is not satisfied, then it is either *violated* ($m \Vdash_v i$) or *avoided* ($m \Vdash_a i$). Specifically:

<u>Truth of a declarative sentence on an interpretation</u>
(C1) $m \vDash e$ iff $e \in \mathbb{S}$.
(C2) $m \vDash \ulcorner \sim p \urcorner$ iff $m \nvDash p$.
(C3) $m \vDash \ulcorner p \ \& \ q \urcorner$ iff both $m \vDash p$ and $m \vDash q$.
(C4) $m \vDash \ulcorner p \vee q \urcorner$ iff either $m \vDash p$ or $m \vDash q$ (or both).
(C5) $m \vDash \ulcorner p \rightarrow q \urcorner$ iff either $m \nvDash p$ or $m \vDash q$.
(C6) $m \vDash \ulcorner p \leftrightarrow q \urcorner$ iff either both $m \vDash p$ and $m \vDash q$ or both $m \nvDash p$ and $m \nvDash q$.

<u>Satisfaction, violation, and avoidance of an imperative sentence on an interpretation</u>
(C7) $m \Vdash_s \ulcorner !p \urcorner$ iff $m \vDash p$, and $m \Vdash_v \ulcorner !p \urcorner$ iff $m \nvDash p$.
(C8) $m \Vdash_s \ulcorner \sim i \urcorner$ iff $m \Vdash_v i$, and $m \Vdash_v \ulcorner \sim i \urcorner$ iff $m \Vdash_s i$.
(C9) $m \Vdash_s \ulcorner i \ \& \ j \urcorner$ iff either both $m \Vdash_s i$ and $m \nVdash_v j$ or both $m \Vdash_s j$ and $m \nVdash_v i$, and $m \Vdash_v \ulcorner i \ \& \ j \urcorner$ iff either $m \Vdash_v i$ or $m \Vdash_v j$.

(C10) $m \Vdash_s \ulcorner i \vee j \urcorner$ iff either $m \Vdash_s i$ or $m \Vdash_s j$, and $m \Vdash_v \ulcorner i \vee j \urcorner$ iff either both $m \Vdash_v i$ and $m \nVdash_s j$ or both $m \Vdash_v j$ and $m \nVdash_s i$.

(C11) $m \Vdash_s \ulcorner p \rightarrow i \urcorner$ iff both $m \vDash p$ and $m \Vdash_s i$, and $m \Vdash_v \ulcorner p \rightarrow i \urcorner$ iff both $m \vDash p$ and $m \Vdash_v i$.

(C12) $m \Vdash_s \ulcorner i \rightarrow p \urcorner$ iff both $m \nvDash p$ and $m \Vdash_v i$, and $m \Vdash_v \ulcorner i \rightarrow p \urcorner$ iff both $m \nvDash p$ and $m \Vdash_s i$.

(C13) $m \Vdash_s \ulcorner p \leftrightarrow i \urcorner$ iff either both $m \vDash p$ and $m \Vdash_s i$ or both $m \nvDash p$ and $m \Vdash_v i$, and $m \Vdash_v \ulcorner p \leftrightarrow i \urcorner$ iff either both $m \vDash p$ and $m \Vdash_v i$ or both $m \nvDash p$ and $m \Vdash_s i$.

(C14) $m \Vdash_s \ulcorner i \leftrightarrow p \urcorner$ iff $m \Vdash_s \ulcorner p \leftrightarrow i \urcorner$, and $m \Vdash_v \ulcorner i \leftrightarrow p \urcorner$ iff $m \Vdash_v \ulcorner p \leftrightarrow i \urcorner$.

(C15) $m \Vdash_a i$ iff both $m \nVdash_s i$ and $m \nVdash_v i$.

Note that, for any m and i, $m \Vdash_s i$ only if $m \nVdash_v i$. See Vranas 2008: 532–45 for a detailed defense of C7–C15. A *contradiction* is either a declarative sentence that is false (i.e., not true) on every interpretation or an imperative sentence that is violated on every interpretation. Sentences φ and ψ are *logically equivalent* (i.e., $\varphi \Leftrightarrow \psi$) only if either they are both declarative or they are both imperative. For declarative sentences p and q, $p \Leftrightarrow q$ iff, for any m, $m \vDash p$ iff $m \vDash q$. (Equivalently, $p \Leftrightarrow q$ iff p and q are interderivable in classical sentential logic.) For imperative sentences i and j, $i \Leftrightarrow j$ iff, for any m, both (1) $m \Vdash_s i$ iff $m \Vdash_s j$ and (2) $m \Vdash_v i$ iff $m \Vdash_v j$.

<u>THEOREM 1 (SEMANTIC REPLACEMENT)</u>. For any imperative sentence i and any sentences φ and ψ, if φ is a subsentence of i and $\varphi \Leftrightarrow \psi$, then $i \Leftrightarrow i(\varphi/\psi)$—where $i(\varphi/\psi)$ is any sentence that results from replacing in i at least one occurrence of φ with ψ.

<u>PROOF</u>. The proof is by induction on the number of occurrences of connectives in i. For the base step, take any i in which no connectives occur. Then, for some e, i is $\ulcorner !e \urcorner$. So if, for some p, $e \Leftrightarrow p$, then $i(e/p)$, namely $\ulcorner !p \urcorner$, is logically equivalent to $\ulcorner !e \urcorner$: for any m, $m \Vdash_s \ulcorner !p \urcorner$ iff $m \vDash p$ iff $m \vDash e$ iff $m \Vdash_s \ulcorner !e \urcorner$ (and similarly $m \Vdash_v \ulcorner !p \urcorner$ iff $m \Vdash_v \ulcorner !e \urcorner$). For the inductive step, take any natural number n and suppose (*induction hypothesis*) that, for any i with at most n occurrences of connectives, and any φ and ψ such that φ is a subsentence of i and $\varphi \Leftrightarrow \psi$, $i \Leftrightarrow i(\varphi/\psi)$. To complete the proof, take any i with at most $n + 1$ occurrences of connectives and any φ and ψ such that φ is a *proper* subsentence of i (the case in which φ is i is trivial) and $\varphi \Leftrightarrow \psi$. To prove that $i \Leftrightarrow i(\varphi/\psi)$, there are eight cases to consider.

Case 1: i is $\ulcorner !p \urcorner$. Then φ is a subsentence of p, and $i(\varphi/\psi)$ is $\ulcorner !p(\varphi/\psi) \urcorner$. By classical sentential logic, $p(\varphi/\psi) \Leftrightarrow p$. It follows, similarly to the base case, that $i \Leftrightarrow i(\varphi/\psi)$.

Case 2: i is $\ulcorner \sim j \urcorner$. Then φ is a subsentence of j, and $i(\varphi/\psi)$ is $\ulcorner \sim j(\varphi/\psi) \urcorner$. By the induction hypothesis, $j \Leftrightarrow j(\varphi/\psi)$ (because j has at most n occurrences of connectives). It follows that $i \Leftrightarrow i(\varphi/\psi)$: for any m, $m \Vdash_s i$ iff $m \Vdash_v j$ iff $m \Vdash_v j(\varphi/\psi)$ iff $m \Vdash_s \ulcorner \sim j(\varphi/\psi) \urcorner$ iff $m \Vdash_s i(\varphi/\psi)$ (and similarly $m \Vdash_v i$ iff $m \Vdash_v i(\varphi/\psi)$).

Case 3: i is $\ulcorner j \ \& \ k \urcorner$. Then φ is a subsentence of j or of k (or both). Suppose it is only of j (if it is only of k, or of both j and k, the proof proceeds similarly). Then $i(\varphi/\psi)$ is $\ulcorner j(\varphi/\psi) \ \& \ k \urcorner$. By the induction hypothesis, $j \Leftrightarrow j(\varphi/\psi)$ (because j has at most n occurrences of connectives). It follows that $i \Leftrightarrow i(\varphi/\psi)$: for any m, $m \Vdash_s i$ iff (either both $m \Vdash_s j$ and $m \nVdash_v k$ or both $m \Vdash_s k$ and $m \nVdash_v j$) iff (either both $m \Vdash_s j(\varphi/\psi)$ and $m \nVdash_v k$ or both $m \Vdash_s k$ and $m \nVdash_v j(\varphi/\psi)$) iff $m \Vdash_s \ulcorner j(\varphi/\psi) \ \& \ k \urcorner$ iff $m \Vdash_s i(\varphi/\psi)$ (and similarly $m \Vdash_v i$ iff $m \Vdash_v i(\varphi/\psi)$).

The proof proceeds similarly in the remaining five cases, namely the cases in which i is $\ulcorner p \vee j \urcorner$, $\ulcorner p \rightarrow j \urcorner$, $\ulcorner j \rightarrow p \urcorner$, $\ulcorner p \leftrightarrow j \urcorner$, or $\ulcorner j \leftrightarrow p \urcorner$, so I omit those cases for the sake of brevity.

4. Replacement interderivability

In this section, I define replacement derivations, and I prove that there is a replacement derivation of j from i iff $i \Leftrightarrow j$.

DEFINITION 1. For any imperative sentences i and j:

(1) A *replacement derivation* of j from i is a finite sequence of imperative sentences (called the *lines* of the derivation) such that (a) the last line is j, (b) the first line is i, and (c) each line except the first can be obtained from the previous line by applying once a replacement rule from Table 1.

(2) i and j are *replacement interderivable* (i.e., $i \dashv\Vdash j$) iff there is a replacement derivation of j from i.

Name of rule and abbreviation		Rule		
Declarative Replacement	DR	If $p \dashv\vdash_{CSL} q$,	then	$p \bowtie q$
Transposition	TR	$i \rightarrow p$	$\bowtie$	$\sim p \rightarrow \sim i$
Negated Conditional	NC	$\sim(p \rightarrow i)$	$\bowtie$	$p \rightarrow \sim i$
Exportation	EX	$p \rightarrow (q \rightarrow i)$	$\bowtie$	$(p \,\&\, q) \rightarrow i$
Commutativity	CO	$p \leftrightarrow i$	$\bowtie$	$i \leftrightarrow p$
Material Equivalence	ME	$p \leftrightarrow i$	$\bowtie$	$(p \rightarrow i) \,\&\, (i \rightarrow p)$
Absorption	AB	$p \rightarrow\, !q$	$\bowtie$	$p \rightarrow\, !(p \,\&\, q)$
Tautologous Antecedent	TA	$(p \vee \sim p) \rightarrow i$	$\bowtie$	i
Unconditional Negation	UN	$\sim !p$	$\bowtie$	$!\sim p$
Imperative Conjunction	IC	$(p \rightarrow\, !q) \,\&\, (p' \rightarrow\, !q')$	$\bowtie$	$(p \vee p') \rightarrow\, !((p \rightarrow q) \,\&\, (p' \rightarrow q'))$
Imperative Disjunction	ID	$(p \rightarrow\, !q) \vee (p' \rightarrow\, !q')$	$\bowtie$	$(p \vee p') \rightarrow\, !((p \,\&\, q) \vee (p' \,\&\, q'))$

Table 1. Replacement rules.

In Table 1, and in what follows, '$p \dashv\vdash_{CSL} q$' abbreviates "p and q are interderivable in classical sentential logic", and for any sentences φ and ψ, '$\varphi \bowtie \psi$' abbreviates "from any imperative sentence k, one can obtain $k(\varphi/\psi)$ if φ is a subsentence of k, and one can obtain $k(\psi/\varphi)$ if ψ is a subsentence of k". For simplicity, I omit corner quotes in tables.

THEOREM 2 (SOUNDNESS OF REPLACEMENT RULES). For any imperative sentences i and j, if $i \bowtie j$ according to a replacement rule in Table 1, then $i \Leftrightarrow j$.

PROOF. For the sake of brevity, I examine only EX, ME, and IC; the proof is similar for the other replacement rules.

Exportation: For any m, $m \Vdash_S \ulcorner p \rightarrow (q \rightarrow i)\urcorner$ iff—by C11—(both $m \vDash p$ and $m \Vdash_S \ulcorner q \rightarrow i\urcorner$) iff ($m \vDash p$, $m \vDash q$, and $m \Vdash_S i$) iff—by C3—(both $m \vDash \ulcorner p \,\&\, q\urcorner$ and $m \Vdash_S i$) iff $m \Vdash_S \ulcorner(p \,\&\, q) \rightarrow i\urcorner$ (and similarly $m \Vdash_V \ulcorner p \rightarrow (q \rightarrow i)\urcorner$ iff $m \Vdash_V \ulcorner(p \,\&\, q) \rightarrow i\urcorner$).

Material Equivalence: Note first that, (1) if $m \Vdash_S \ulcorner p \rightarrow i\urcorner$ (i.e., both $m \vDash p$ and $m \Vdash_S i$), then $m \nVdash_V \ulcorner i \rightarrow p\urcorner$ (i.e., it is not the case that both $m \nvDash p$ and $m \Vdash_S i$). Similarly, (2) if $m \Vdash_S \ulcorner i \rightarrow p\urcorner$, then $m \nVdash_V \ulcorner p \rightarrow i\urcorner$. Now, for any m: $m \Vdash_S \ulcorner p \leftrightarrow i\urcorner$ iff—by C13—(either both $m \vDash p$ and $m \Vdash_S i$ or both $m \nvDash p$ and $m \Vdash_V i$) iff—by C11 and C12—(either $m \Vdash_S \ulcorner p \rightarrow i\urcorner$ or $m \Vdash_S \ulcorner i \rightarrow p\urcorner$) iff—by (1) and (2)—(either both $m \Vdash_S \ulcorner p \rightarrow i\urcorner$ and $m \nVdash_V \ulcorner i \rightarrow p\urcorner$ or both $m \Vdash_S \ulcorner i \rightarrow p\urcorner$ and $m \nVdash_V \ulcorner p \rightarrow i\urcorner$) iff—by C9—$m \Vdash_S \ulcorner(p \rightarrow i) \,\&\, (i \rightarrow p)\urcorner$ (and similarly $m \Vdash_V \ulcorner p \leftrightarrow i\urcorner$ iff $m \Vdash_V \ulcorner(p \rightarrow i) \,\&\, (i \rightarrow p)\urcorner$).

Imperative Conjunction: Note first that $m \Vdash_S \ulcorner p \to !q \urcorner$ iff (both $m \vDash p$ and $m \Vdash_S !q$) iff (both $m \vDash p$ and $m \vDash q$) iff $m \vDash \ulcorner p \,\&\, q \urcorner$. Similarly, $m \Vdash_V \ulcorner p \to !q \urcorner$ iff $m \vDash \ulcorner p \,\&\, {\sim}q \urcorner$. Now, for any m: $m \Vdash_S \ulcorner (p \to !q) \,\&\, (p' \to !q') \urcorner$ iff—by C9—(either both $m \Vdash_S \ulcorner p \to !q \urcorner$ and $m \nVdash_V \ulcorner p' \to !q' \urcorner$ or both $m \Vdash_S \ulcorner p' \to !q' \urcorner$ and $m \nVdash_V \ulcorner p \to !q \urcorner$) iff (either both $m \vDash \ulcorner p \,\&\, q \urcorner$ and $m \nvDash \ulcorner p' \,\&\, {\sim}q' \urcorner$ or both $m \vDash \ulcorner p' \,\&\, q' \urcorner$ and $m \nvDash \ulcorner p \,\&\, {\sim}q \urcorner$) iff $m \vDash \ulcorner ((p \,\&\, q) \,\&\, {\sim}(p' \,\&\, {\sim}\, q')) \vee ((p' \,\&\, q') \,\&\, {\sim}(p \,\&\, {\sim}q)) \urcorner$ iff—by classical sentential logic—$m \vDash \ulcorner (p \vee p') \,\&\, ((p \to q) \,\&\, (p' \to q')) \urcorner$ iff $m \Vdash_S \ulcorner (p \vee p') \to !((p \to q) \,\&\, (p' \to q')) \urcorner$ (and similarly for violation).

<u>THEOREM 3 (SYNTACTIC REPLACEMENT)</u>. For any imperative sentences i, j, and k, if j is a subsentence of i and $j \dashv\vdash k$, then $i \dashv\vdash i(j/k)$.

<u>PROOF</u>. Suppose $j \dashv\vdash k$. The proof is by induction on the number of lines of a replacement derivation. For the base step, suppose there is a one-line replacement derivation of k from j. Then j is the same sentence as k and thus $i \dashv\vdash i(j/k)$. For the inductive step, take any non-zero natural number n and suppose (*induction hypothesis*) that, if there is a replacement derivation with n lines of k from j, then $i \dashv\vdash i(j/k)$. To complete the proof, take any replacement derivation with n + 1 lines of k from j. Then k can be obtained from the n-th line k' by applying once a replacement rule, so k is $k'(\varphi/\psi)$, where φ is a subsentence of k' and ψ is a sentence such that $\varphi \bowtie \psi$. Let i' be the sentence that results from replacing with k' in i exactly those occurrences of j that are replaced with k in i to get $i(j/k)$. By the induction hypothesis, (1) $i \dashv\vdash i'$. Since k is $k'(\varphi/\psi)$, $i(j/k)$ results from replacing in i' some occurrences of φ with ψ. So $i(j/k)$ is $i'(\varphi/\psi)$, and thus—since $\varphi \bowtie \psi$—(2) $i(j/k)$ can be obtained from i' by applying once a replacement rule. By (1) and (2), $i \dashv\vdash i(j/k)$.

<u>THEOREM 4 (CANONICAL FORM)</u>. For any imperative sentence i, there are declarative sentences p and q such that $i \dashv\vdash \ulcorner p \to !q \urcorner$.

<u>PROOF</u>. The proof is by induction on the number of occurrences of connectives in i. For the base step, take any i in which no connectives occur. Then, for some e, i is $\ulcorner !e \urcorner$, and then, by TA (see Table 1), $i \dashv\vdash \ulcorner (e \vee {\sim}e) \to !e \urcorner$. For the inductive step, take any natural number n and suppose (*induction hypothesis*) that, for any i with at most n occurrences of connectives, there are p and q such that $i \dashv\vdash \ulcorner p \to !q \urcorner$. To complete the proof, take any i with at most n + 1 occurrences of connectives. There are eight cases to consider.

Case 1: i is $\ulcorner !p \urcorner$. Then, by TA, $i \dashv\vdash \ulcorner (p \vee {\sim}p) \to !p \urcorner$.

Case 2: i is $\ulcorner {\sim}j \urcorner$. Then j has at most n occurrences of connectives and thus, by the induction hypothesis, $j \dashv\vdash \ulcorner p \to !q \urcorner$ (for some p and q; I omit such remarks in what follows). Then, by Theorem 3, $i \dashv\vdash \ulcorner {\sim}(p \to !q) \urcorner$, and then, by NC and UN, $i \dashv\vdash \ulcorner p \to !{\sim}q \urcorner$.

Case 3: i is $\ulcorner j \,\&\, k \urcorner$. Then j has most n occurrences of connectives and thus, by the induction hypothesis, $j \dashv\vdash \ulcorner p \to !q \urcorner$. Similarly, $k \dashv\vdash \ulcorner p' \to !q' \urcorner$. So, by Theorem 3, $i \dashv\vdash \ulcorner (p \to !q) \,\&\, (p' \to !q') \urcorner$, and thus, by IC, $i \dashv\vdash \ulcorner (p \vee p') \to !((p \to q) \,\&\, (p' \to q')) \urcorner$.

Case 4: i is $\ulcorner j \vee k \urcorner$. Then, similarly to case 3, $i \dashv\vdash \ulcorner (p \to !q) \vee (p' \to !q') \urcorner$, and thus, by ID, $i \dashv\vdash \ulcorner (p \vee p') \to !((p \,\&\, q) \vee (p' \,\&\, q')) \urcorner$.

Case 5: i is $\ulcorner p \to j \urcorner$. Then, by the induction hypothesis, $j \dashv\vdash \ulcorner q \to !r \urcorner$. So, by Theorem 3, $i \dashv\vdash \ulcorner p \to (q \to !r) \urcorner$, and thus, by EX, $i \dashv\vdash \ulcorner (p \,\&\, q) \to !r \urcorner$.

Case 6: i is $\ulcorner j \to p \urcorner$. Then, similarly to case 5, $i \dashv\vdash \ulcorner (q \to !r) \to p \urcorner$, and thus, by TR, NC, EX, and UN, $i \dashv\vdash \ulcorner (\sim p \ \& \ q) \to !\sim r \urcorner$.

Case 7: i is $\ulcorner p \leftrightarrow j \urcorner$. Then, similarly to case 5, $i \dashv\vdash \ulcorner p \leftrightarrow (q \to !r) \urcorner$, and thus, by ME, $i \dashv\vdash \ulcorner (p \to (q \to !r)) \ \& \ ((q \to !r) \to p) \urcorner$. So, by the replacement rules used in case 6, $i \dashv\vdash \ulcorner ((p \ \& \ q) \to !r) \ \& \ ((\sim p \ \& \ q) \to !\sim r) \urcorner$, and thus, by IC, $i \dashv\vdash \ulcorner ((p \ \& \ q) \vee (\sim p \ \& \ q)) \to !(((p \ \& \ q) \to r) \ \& \ ((\sim p \ \& \ q) \to \sim r)) \urcorner$.

Case 8: i is $\ulcorner j \leftrightarrow p \urcorner$. Then, by CO, $i \dashv\vdash \ulcorner p \leftrightarrow j \urcorner$, and the proof proceeds as in case 7.

<u>Theorem 5 (Soundness and Completeness for Replacement Interderivability)</u>. For any imperative sentences i and j, $i \Leftrightarrow j$ if (*soundness*) and only if (*completeness*) $i \dashv\vdash j$.

<u>Proof of Soundness</u>. Suppose $i \dashv\vdash j$. The proof is by induction on the number of lines of a replacement derivation. For the base step, suppose there is a one-line replacement derivation of j from i. Then i is the same sentence as j and thus $i \Leftrightarrow j$. For the inductive step, take any non-zero natural number n and suppose (*induction hypothesis*) that, if there is a replacement derivation with n lines of j from i, then $i \Leftrightarrow j$. To complete the proof, take any replacement derivation with $n + 1$ lines of j from i. Then j can be obtained from the n-th line k by applying once a replacement rule, so j is $k(\varphi/\psi)$, where φ is a subsentence of k and ψ is a sentence such that $\varphi \bowtie \psi$. By the induction hypothesis, (1) $i \Leftrightarrow k$. By Theorem 2, $\varphi \Leftrightarrow \psi$ if φ and ψ are imperative sentences; if they are declarative, then j can be obtained from k by applying once DR, so $\varphi \dashv\vdash_{CSL} \psi$ and thus again $\varphi \Leftrightarrow \psi$. By Theorem 1, $k \Leftrightarrow k(\varphi/\psi)$; i.e., (2) $k \Leftrightarrow j$. By (1), (2), and the transitivity of logical equivalence (which follows from its definition in §3), $i \Leftrightarrow j$.

<u>Proof of Completeness</u>. Suppose $i \Leftrightarrow j$. By Theorem 4, there are p, q, p', and q' such that (1) $i \dashv\vdash \ulcorner p \to !q \urcorner$ and thus (by soundness) $i \Leftrightarrow \ulcorner p \to !q \urcorner$, and (2) $j \dashv\vdash \ulcorner p' \to !q' \urcorner$ and thus $j \Leftrightarrow \ulcorner p' \to !q' \urcorner$. Then (3) $\ulcorner p \to !q \urcorner \Leftrightarrow \ulcorner p' \to !q' \urcorner$. It follows that $p \Leftrightarrow p'$: for any m, $m \vDash p$ iff (either both $m \vDash p$ and $m \vDash q$ or both $m \vDash p$ and $m \nvDash q$) iff—by C11—(either $m \Vdash_S \ulcorner p \to !q \urcorner$ or $m \Vdash_V \ulcorner p \to !q \urcorner$) iff—by (3)—(either $m \Vdash_S \ulcorner p' \to !q' \urcorner$ or $m \Vdash_V \ulcorner p' \to !q' \urcorner$) iff (either both $m \vDash p'$ and $m \vDash q'$ or both $m \vDash p'$ and $m \nvDash q'$) iff $m \vDash p'$. Since $p \Leftrightarrow p'$, (4) $p \dashv\vdash_{CSL} p'$. One can show similarly that $\ulcorner p \ \& \ q \urcorner \Leftrightarrow \ulcorner p' \ \& \ q' \urcorner$, so (5) $\ulcorner p \ \& \ q \urcorner \dashv\vdash_{CSL} \ulcorner p' \ \& \ q' \urcorner$. To conclude: i is replacement interderivable, by (1), with $\ulcorner p \to !q \urcorner$, and thus also, by AB, with $\ulcorner p \to !(p \ \& \ q) \urcorner$, and thus also, by (4) and DR, with $\ulcorner p' \to !(p \ \& \ q) \urcorner$, and thus also, by (5) and DR, with $\ulcorner p' \to !(p' \ \& \ q') \urcorner$, and thus also, by AB, with $\ulcorner p' \to !q' \urcorner$, and thus finally, by (2), with j.

<u>Corollary 1 (of Theorems 4 and 5)</u>. For any imperative sentence i, there are declarative sentences p and q such that $i \Leftrightarrow \ulcorner p \to !q \urcorner$.

<u>Corollary 2 (of Theorems 4 and 5)</u>. For any imperative sentence i, there are declarative sentences s and v such that, for any m, $m \vDash s$ iff $m \Vdash_S i$ and $m \vDash v$ iff $m \Vdash_V i$. (<u>Proof</u>. By Corollary 1, there are p and q such that $i \Leftrightarrow \ulcorner p \to !q \urcorner$. Then, for any m, $m \Vdash_S i$ iff $m \Vdash_S \ulcorner p \to !q \urcorner$ iff (by C11, C7, and C3) $m \vDash \ulcorner p \ \& \ q \urcorner$, so take s to be $\ulcorner p \ \& \ q \urcorner$. Similarly, take v to be $\ulcorner p \ \& \sim q \urcorner$.)

5. Strong and weak semantic validity

A *pure imperative argument* (of the language of SPIL) is an ordered pair $\langle \Gamma, i \rangle$, where Γ is a non-empty finite set of imperative sentences (the *premises* of the argument) and i is an imperative sentence (the *conclusion* of the argument). In this paper, I do not examine arguments whose premises and conclusions include both declarative and imperative sentences (e.g., the argument

$\langle\{A \rightarrow !B, A\}, !B\rangle$). Building on previous work (Vranas 2011, 2016), I say that (roughly) a pure imperative argument is semantically valid when, on every interpretation, its conclusion is "supported" by everything that supports its premises. Also building on previous work, I distinguish *strong* from *weak* support—and, correspondingly, strong from weak semantic validity—as follows:

DEFINITION 2. For any declarative sentence p, any imperative sentence i, and any interpretation m:
(1) *p strongly supports i on m* iff (a) $m \vDash p$, (b) i is not a contradiction, and (c) $\langle p, q, r\rangle \in \mathbb{F}$ for any q and r that are not both contradictions and are such that, for any m', both (i) $m' \vDash q$ only if $m' \Vdash_s i$ and (ii) $m' \vDash r$ only if $m' \Vdash_v i$.
(2) *p weakly supports i on m* iff p strongly supports on m some j such that, for any m', both (a) $m' \Vdash_s j$ only if $m' \Vdash_s i$ and (b) $m' \Vdash_a i$ iff $m' \Vdash_a j$.

DEFINITION 3. A pure imperative argument $\langle\Gamma, i\rangle$ is (1) *strongly semantically valid* (i.e., $\Gamma \Vdash_s i$) iff, for any m, every p that *strongly* supports on m every conjunction[2] of all members of Γ also *strongly* supports i on m, and is (2) *weakly semantically valid* (i.e., $\Gamma \Vdash_w i$) iff, for any m, every p that *weakly* supports on m every conjunction of all members of Γ also *weakly* supports i on m.

It follows from Definition 2 that, if p strongly supports i on m, then p also weakly supports i on m. Informally, the distinction between strong and weak semantic validity captures a conflict of intuitions about whether, for example, "sign the letter" entails "sign or burn the letter": one can show that the pure imperative argument $\langle\{!S\}, !(S \vee B)\rangle$ is weakly but not strongly semantically valid.[3]

THEOREM 6 (SEMANTIC EQUIVALENCE). For any imperative sentences i and j:
(1) $i \Vdash_s j$ (i.e., $\{i\} \Vdash_s j$) iff either i is a contradiction or, for any m, both (a) $m \Vdash_s j$ only if $m \Vdash_s i$ and (b) $m \Vdash_v j$ only if $m \Vdash_v i$;
(2) $i \Vdash_w j$ iff, for any m, both (a) $m \Vdash_a i$ only if $m \Vdash_a j$ and (b) $m \Vdash_v j$ only if $m \Vdash_v i$.

2 See Vranas 2011: 396–8 for an explanation of why I define semantic validity in terms of supporting *conjunctions* of all premises and not in terms of supporting *every* premise. Given the intensionality condition (§3) and the logical equivalence of any two conjunctions of all premises of an argument, supporting (strongly or weakly, on an interpretation) *some* conjunction of all premises of an argument amounts to supporting *every* conjunction of all premises of the argument. Because sentences are *finite* strings of symbols, I do not define conjunctions of infinitely many sentences (contrast Vranas 2016: 1706 n. 1); this is why I defined an argument as having finitely many premises.

3 Defending the above definitions lies beyond the scope of this paper: I have extensively defended in previous work (Vranas 2011, 2016) an account of validity on which the definitions are based. I say that the definitions are "based" on my previously defended account of validity because that account is about "arguments" whose premises and conclusions are not sentences of a formal language, but are instead what imperative and declarative sentences of natural languages typically express, namely *prescriptions* (i.e., commands, requests, instructions, suggestions, etc.) and *propositions* respectively. Deviating slightly from previous work in order to keep my definition of an interpretation (§3) simple, I formulated Definition 2 so that it has as consequences two claims corresponding to what in previous work I understood as assumptions about favoring, namely the claims that (1) no declarative sentence strongly supports on any interpretation an imperative sentence which is a contradiction (cf. Assumption 1 in Vranas 2011: 433) and (2) every declarative sentence that is true on an interpretation strongly supports on that interpretation any *semantically empty* imperative sentence (cf. Vranas 2016: 1708 n. 6), namely any imperative sentence that is avoided on every interpretation.

<u>PROOF</u>. The theorem provides necessary and sufficient conditions for strong and for weak semantic validity. The proof has four parts, and is similar to the proof in Appendix A of Vranas 2011.

First part: Sufficient condition for strong semantic validity. If i is a contradiction, then (by Definition 2) no p strongly supports i on any m, and then (by Definition 3) $i \Vdash_S j$. If, for any m', both (a) $m' \VDash_S j$ only if $m' \VDash_S i$ and (b) $m' \VDash_V j$ only if $m' \VDash_V i$, take any $m = \langle \mathbb{S}, \mathbb{F} \rangle$ and any p. Suppose that (1) p strongly supports i on m. Then (2) $m \vDash p$ (by Definition 2) and (3) j is not a contradiction (because, by Definition 2, i is not a contradiction; so, for some m', $m' \nVDash_V i$, and thus—by (b)— $m' \nVDash_V j$). Moreover, (4) for any q and r, if q and r are not both contradictions and are such that, for any m', both (i) $m' \vDash q$ only if $m' \VDash_S j$ and (ii) $m' \vDash r$ only if $m' \VDash_V j$, then $\langle p, q, r \rangle \in \mathbb{F}$ (by (1) and Definition 2, because (by (i) and (a)) $m' \vDash q$ only if $m' \VDash_S i$ and (by (ii) and (b)) $m' \vDash r$ only if $m' \VDash_V i$). By (2), (3), (4), and Definition 2, p strongly supports j on m, so (by Definition 3) $i \Vdash_S j$.

Second part: Necessary condition for strong semantic validity. By Corollary 2, there are declarative sentences s and v such that, for any m, $m \vDash s$ iff $m \VDash_S i$ and $m \vDash v$ iff $m \VDash_V i$, and declarative sentences s' and v' that satisfy the corresponding conditions with respect to j. Suppose, for reductio, that (1) $i \Vdash_S j$ but (2) i is not a contradiction and (3) it is not the case that, for every m, both (a) $m \VDash_S j$ only if $m \VDash_S i$ and (b) $m \VDash_V j$ only if $m \VDash_V i$. Consider an interpretation $m = \langle \mathbb{S}, \mathbb{F} \rangle$, where $\mathbb{S} = \{e\}$ for some e (so (4) $m \vDash e$) and $\mathbb{F}$ is the set of ordered triples $\langle p, q, r \rangle$ such that (i) $p \Leftrightarrow e$, (ii) $q \vdash_{CSL} s$ (i.e., s is derivable from q in classical sentential logic; equivalently, for any m', $m' \vDash q$ only if $m' \VDash_S i$), (iii) $r \vdash_{CSL} v$ (equivalently, for any m', $m' \vDash r$ only if $m' \VDash_V i$), and (iv) q and r are not both contradictions. $\mathbb{F}$ satisfies the asymmetry condition (§3): if one supposes for reductio that both $\langle p, q, r \rangle \in \mathbb{F}$ and $\langle p, r, q \rangle \in \mathbb{F}$, then one gets that q and r are both contradictions (contradicting (iv)): q is a contradiction because, for any m', if $m' \vDash q$, then both $m' \VDash_S i$ and $m' \VDash_V i$ (which is impossible), and similarly for r. The intensionality condition (§3) is also satisfied. By (2), (4), the definition of $\mathbb{F}$, and Definition 2, e strongly supports i on m. Then, by (1) and Definition 3, (5) e also strongly supports j on m. Let q be $\ulcorner s' \,\&\, {\sim}s \urcorner$ and r be $\ulcorner v' \,\&\, {\sim}v \urcorner$. By (3), q and r are not both contradictions. Moreover, for any m', $m' \vDash q$ only if $m' \VDash_S j$, and $m' \vDash r$ only if $m' \VDash_V j$. Then, by (5) and Definition 2, $\langle e, q, r \rangle \in \mathbb{F}$. By (ii), $\ulcorner s' \,\&\, {\sim}s \urcorner \vdash_{CSL} s$, so (there is no interpretation on which $\ulcorner (s' \,\&\, {\sim}s) \,\&\, {\sim}s \urcorner$ is true, and thus) $s' \vdash_{CSL} s$; equivalently, (6) for any m, $m \VDash_S j$ only if $m \VDash_S i$. Similarly, by (iii), $\ulcorner v' \,\&\, {\sim}v \urcorner \vdash_{CSL} v$, so $v' \vdash_{CSL} v$; equivalently, (7) for any m, $m \VDash_V j$ only if $m \VDash_V i$. But (6) and (7) together contradict (3), and the reductio is complete.

Third part: Sufficient condition for weak semantic validity. Suppose that, for any m, both (a) $m \VDash_A i$ only if $m \VDash_A j$ and (b) $m \VDash_V j$ only if $m \VDash_V i$. Take any m and any p that weakly supports i on m. By Definition 2, (1) p strongly supports on m some imperative sentence i^* such that, for any m', both (i) $m' \VDash_S i^*$ only if $m' \VDash_S i$ and (ii) $m' \VDash_A i$ iff $m' \VDash_A i^*$. Let k be $\ulcorner (s' \vee v') \rightarrow\, !(s^* \,\&\, s') \urcorner$, where s' and v' are as in the second part of the proof and s^* is a declarative sentence such that, for any m', $m' \vDash s^*$ iff $m' \VDash_S i^*$ (see Corollary 2). Then, (2) for any m', $m' \VDash_S k$ only if $m' \vDash \ulcorner s^* \,\&\, s' \urcorner$, and thus, (3) for any m', $m' \VDash_S k$ only if $m' \VDash_S i^*$. Moreover, (4) for any m', $m' \VDash_V k$ only if $m' \VDash_V i^*$ (as one can show by using (a), (b), (i), and (ii); see Vranas 2011: 436 n. 68). By (3), (4), and the first part of the proof, (5) $i^* \Vdash_S k$. By (1), (5), and Definition 3, (6) p strongly supports k on m. But, (7) for any m', $m' \VDash_S k$ only if $m' \VDash_S j$ (by (2)), and, (8) for any m', $m' \VDash_A j$

iff $m' \VDash_a k$ (because $m' \VDash_a k$ iff $m' \nvDash \ulcorner s' \vee v' \urcorner$). By (6), (7), (8), and Definition 2, p weakly supports j on m, so (by Definition 3) $i \Vdash_w j$.

Fourth part: Necessary condition for weak semantic validity. Suppose, for reductio, that (1) $i \Vdash_w j$ but (2) either (a) for some m, both $m \VDash_a i$ and $m \nVDash_a j$, or (b) for some m, both $m \VDash_v j$ and $m \nVDash_v i$ (i.e., it is not the case that, for every m, both (a′) $m \VDash_a i$ only if $m \VDash_a j$ and (b′) $m \VDash_v j$ only if $m \VDash_v i$). By (2), i is not a contradiction (i.e., for some m, $m \nVDash_v i$; this is immediate if (b) is true, and follows from $m \VDash_a i$ if (a) is true). Consider an interpretation $m = \langle \mathbb{S}, \mathbb{F} \rangle$ defined as in the second part of the proof. As in that part, e strongly supports i on m, so e also weakly supports i on m. Then, by (1) and Definition 3, e also weakly supports j on m. Then, by Definition 2, (3) e strongly supports on m some i^* such that, for any m', both (i) $m' \VDash_s i^*$ only if $m' \VDash_s j$ and (ii) $m' \VDash_a j$ iff $m' \VDash_a i^*$. By (2), for some m, $m \nVDash_a j$ (this is immediate if (a) is true, and follows from $m \VDash_v j$ if (b) is true). Then, by (ii), for some m, $m \nVDash_a i^*$, so s^* and v^* are not both contradictions—where s^* is as in the third part of the proof and v^* is a declarative sentence such that, for any m, $m \vDash v^*$ iff $m \VDash_v i^*$ (see Corollary 2). Then, by (3) and Definition 2, $\langle e, s^*, v^* \rangle \in \mathbb{F}$, and by the definition of $\mathbb{F}$ in the second part of the proof, (4) $s^* \vdash_{CSL} s$ and (5) $v^* \vdash_{CSL} v$. But then (a) is false: for any m, if $m \nVDash_a j$ and thus (by (ii)) $m \nVDash_a i^*$, then $m \nVDash_a i$ (because either $m \VDash_s i^*$, and then by (4) $m \VDash_s i$ and thus $m \nVDash_a i$, or $m \VDash_v i^*$, and then by (5) $m \VDash_v i$ and thus $m \nVDash_a i$). Moreover, (b) is false: for any m, if $m \VDash_v j$ (and thus (6) $m \nVDash_s j$ and (7) $m \nVDash_a j$), then $m \VDash_v i$ (because $m \nVDash_s i^*$, by (i) and (6), and $m \nVDash_a i^*$, by (ii) and (7), so $m \VDash_v i^*$ and, by (5), $m \VDash_v i$). The falsity of (a) and (b) contradicts (2), and the reductio is complete.

<u>Corollary 3 (Of Theorem 6)</u>. For any imperative sentences i and j, (1) $i \Vdash_s j$ only if $i \Vdash_w j$, and (2) $i \Leftrightarrow j$ iff (both $i \Vdash_s j$ and $j \Vdash_s i$) iff (both $i \Vdash_w j$ and $j \Vdash_w i$).

6. Strong and weak derivability

In this section, I define strong and weak derivations, and I prove that there is a strong (or weak) derivation of i from Γ iff the argument $\langle \Gamma, i \rangle$ is strongly (or weakly) semantically valid.

<u>Definition 4</u>. For any pure imperative argument $\langle \Gamma, i \rangle$:

(1) A *strong derivation* of i from Γ is a finite sequence of imperative sentences (called the *lines* of the derivation) such that (a) the last line is i and (b) each line either is a conjunction of *all* members of Γ or can be obtained from a previous line by applying once either a replacement rule from Table 1 or a pure imperative inference rule (*other than ICE*) from Table 2.

(2) A *weak derivation* of i from Γ is a finite sequence of imperative sentences (called the *lines* of the derivation) such that (a) the last line is i and (b) each line either is (a member or) a conjunction of members of Γ or can be obtained from a previous line by applying once either a replacement rule from Table 1 or a pure imperative inference rule from Table 2.

(3) $\langle \Gamma, i \rangle$ is (a) *strongly syntactically valid* (i.e., $\Gamma \vdash_s i$) iff there is a strong derivation of i from Γ, and is (b) *weakly syntactically valid* (i.e., $\Gamma \vdash_w i$) iff there is a weak derivation of i from Γ.

Name of rule and abbreviation		**Rule**
Ex Contradictione Quodlibet	ECQ	$!(p \,\&\, {\sim}p) \blacktriangleright i$
Declarative Antecedent Introduction	DAI	$i \blacktriangleright p \rightarrow i$
Imperative Conjunction Elimination	ICE	$i \,\&\, j \blacktriangleright i$

Table 2. Pure imperative inference rules.

In Table 2, and in what follows, for any imperative sentences i and j, '$i \blacktriangleright j$' abbreviates "from i, one can obtain j". It follows from Definition 4 that every strong derivation is a weak derivation, so $\Gamma \vdash_s i$ only if $\Gamma \vdash_w i$. Moreover, since replacement rules may be applied in strong derivations, $j \dashv\vdash i$ only if $j \vdash_s i$ (i.e., $\{j\} \vdash_s i$). Note two differences between weak and strong derivations. First, all pure imperative inference rules in Table 2 may be applied in a weak derivation, but Imperative Conjunction Elimination (ICE) may *not* be applied in a strong derivation. The motivation behind this difference is that, for example, the argument $\langle\{!A \ \&\ !B\}, !A\rangle$ is (weakly but) not strongly semantically valid (as one can show by using Theorem 6), but strong derivations are intended to correspond to strong semantic validity. Second, any single premise can be the first line of a weak derivation, but no single premise (as opposed to a conjunction of all premises) can be the first line of a strong derivation (unless there is only one premise). The motivation behind this difference is that, for example, the argument $\langle\{!A, !B\}, !A\rangle$ is (weakly but) not strongly semantically valid (see Vranas 2011: 397).[4]

<u>THEOREM 7 (SOUNDNESS OF INFERENCE RULES)</u>. For any declarative sentence p and any imperative sentences i and j: (1) $\ulcorner!(p \ \&\ {\sim}p)\urcorner \Vdash_s i$; (2) $i \Vdash_s \ulcorner p \rightarrow i\urcorner$; (3) $\ulcorner i \ \&\ j\urcorner \Vdash_w i$.

<u>PROOF</u>. (1) Since $\ulcorner!(p \ \&\ {\sim}p)\urcorner$ is a contradiction, $\ulcorner!(p \ \&\ {\sim}p)\urcorner \Vdash_s i$ by Theorem 6. (2) For any m, both (a) $m \Vert\!\models_s \ulcorner p \rightarrow i\urcorner$ only if $m \Vert\!\models_s i$ (by C11) and (b) $m \Vert\!\models_v \ulcorner p \rightarrow i\urcorner$ only if $m \Vert\!\models_v i$ (by C11), so $i \Vdash_s \ulcorner p \rightarrow i\urcorner$ by Theorem 6. (3) For any m, both (a) $m \Vert\!\models_a \ulcorner i \ \&\ j\urcorner$ only if $m \Vert\!\models_a i$ (by C9 and C15) and (b) $m \Vert\!\models_v i$ only if $m \Vert\!\models_v \ulcorner i \ \&\ j\urcorner$ (by C9), so $\ulcorner i \ \&\ j\urcorner \Vdash_w i$ by Theorem 6.

<u>THEOREM 8 (STRENGTHENING THE ANTECEDENT AND WEAKENING THE CONSEQUENT)</u>. For any declarative sentences p, p', q, and q', and any imperative sentence i: (1) if $p' \vdash_{CSL} p$, then $\ulcorner p \rightarrow i\urcorner \vdash_s \ulcorner p' \rightarrow i\urcorner$; (2) if $q \vdash_{CSL} q'$, then $\ulcorner p \rightarrow !q\urcorner \vdash_w \ulcorner p \rightarrow !q'\urcorner$.

<u>PROOF</u>. (1) $\ulcorner p \rightarrow i\urcorner \vdash_s \ulcorner p' \rightarrow (p \rightarrow i)\urcorner$ (by DAI), and $\ulcorner p' \rightarrow (p \rightarrow i)\urcorner \vdash_s \ulcorner(p' \ \&\ p) \rightarrow i\urcorner$ (by EX). But if $p' \vdash_{CSL} p$, then $\ulcorner p' \ \&\ p\urcorner \dashv\vdash_{CSL} p'$, and then $\ulcorner(p' \ \&\ p) \rightarrow i\urcorner \vdash_s \ulcorner p' \rightarrow i\urcorner$ (by DR). (2) If $q \vdash_{CSL} q'$, then $\ulcorner q' \ \&\ q\urcorner \dashv\vdash_{CSL} q$. Then there is a weak derivation from $\ulcorner p \rightarrow !q\urcorner$ of $\ulcorner p \rightarrow !(q' \ \&\ q)\urcorner$ (by DR), and thus also of $\ulcorner p \rightarrow !(p \ \&\ (q' \ \&\ q))\urcorner$ (by AB), and thus also of $\ulcorner(p \vee p) \rightarrow !((p \vee p) \ \&\ ((p \rightarrow q') \ \&\ (p \rightarrow q)))\urcorner$ (by DR, since $p \dashv\vdash_{CSL} \ulcorner p \vee p\urcorner$ and $\ulcorner p \ \&\ (q' \ \&\ q)\urcorner \dashv\vdash_{CSL} \ulcorner(p \vee p) \ \&\ ((p \rightarrow q') \ \&\ (p \rightarrow q))\urcorner$), and thus also of $\ulcorner(p \vee p) \rightarrow !((p \rightarrow q') \ \&\ (p \rightarrow q))\urcorner$ (by AB), and thus also of $\ulcorner(p \rightarrow !q') \ \&\ (p \rightarrow !q)\urcorner$ (by IC), and thus finally of $\ulcorner p \rightarrow !q'\urcorner$ (by ICE).

<u>THEOREM 9 (SOUNDNESS AND COMPLETENESS FOR STRONG AND WEAK DERIVABILITY)</u>. For any pure imperative argument $\langle\Gamma, i\rangle$, (1) $\Gamma \Vdash_s i$ if (*soundness*) and only if (*completeness*) $\Gamma \vdash_s i$, and (2) $\Gamma \Vdash_w i$ if (*soundness*) and only if (*completeness*) $\Gamma \vdash_w i$.

<u>PROOF OF SOUNDNESS</u>. The proof is by induction on the number of lines of a strong or weak derivation. For the base step, suppose there is a one-line strong (case 1) or weak (case 2) derivation of i from Γ. In case 1, i is a conjunction of all members of Γ and thus (by Definition 3) $\Gamma \Vdash_s i$. In

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[4] DAI is redundant given ICE, AB, IC, and EX. Indeed, $\ulcorner p \rightarrow i\urcorner$ can be obtained by ICE from $\ulcorner(p \rightarrow i) \ \&\ ({\sim}p \rightarrow i)\urcorner$, which is replacement interderivable with i: i is replacement interderivable with $\ulcorner q \rightarrow !r\urcorner$ (for some q and r, by Theorem 4), and thus also with $\ulcorner q \rightarrow !(q \ \&\ r)\urcorner$ (by AB), and thus also with $\ulcorner q \rightarrow !(q \ \&\ (q \rightarrow r))\urcorner$ (by DR, since $\ulcorner q \ \&\ r\urcorner \dashv\vdash_{CSL} \ulcorner q \ \&\ (q \rightarrow r)\urcorner$), and thus also with $\ulcorner q \rightarrow !(q \rightarrow r)\urcorner$ (by AB), and thus also with $\ulcorner((p \ \&\ q) \vee ({\sim}p \ \&\ q)) \rightarrow !(((p \ \&\ q) \rightarrow r) \ \&\ (({\sim}p \ \&\ q) \rightarrow r))\urcorner$ (by DR, since $q \dashv\vdash_{CSL} \ulcorner(p \ \&\ q) \vee ({\sim}p \ \&\ q)\urcorner$ and $\ulcorner q \rightarrow r\urcorner \dashv\vdash_{CSL} \ulcorner((p \ \&\ q) \rightarrow r) \ \&\ (({\sim}p \ \&\ q) \rightarrow r)\urcorner$), and thus also with $\ulcorner((p \ \&\ q) \rightarrow !r) \ \&\ (({\sim}p \ \&\ q) \rightarrow !r)\urcorner$ (by IC), and thus also with $\ulcorner(p \rightarrow (q \rightarrow !r)) \ \&\ ({\sim}p \rightarrow (q \rightarrow !r))\urcorner$ (by EX), and thus finally with $\ulcorner(p \rightarrow i) \ \&\ ({\sim}p \rightarrow i)\urcorner$ (by Theorem 3). It does not follow, however, that DAI is redundant in *strong* derivations: ICE may *not* be applied in strong derivations.

case 2, i is (a member or) a conjunction of members of Γ; so, if i is not a conjunction of *all* members of Γ (if it is, the proof proceeds as in case 1), there is a conjunction j of the remaining members of Γ, and $\ulcorner i \,\&\, j \urcorner$ is a conjunction of all members of Γ. Then $\Gamma \Vdash_W i$ because, by Definition 3, $\Gamma \Vdash_W \ulcorner i \,\&\, j \urcorner$, and by Theorem 7, $\ulcorner i \,\&\, j \urcorner \Vdash_W i$. For the inductive step, take any non-zero natural number n and suppose (*induction hypothesis*) that: (case 1) if there is a strong derivation with at most n lines of i from Γ, then $\Gamma \Vdash_S i$; (case 2) if there is a weak derivation with at most n lines of i from Γ, then $\Gamma \Vdash_W i$. To complete the proof, take any strong (case 1) or weak (case 2) derivation with at most $n + 1$ lines of i from Γ. Suppose that i is *not* a conjunction of all (case 1) or some (case 2) members of Γ (if it is, the proof proceeds as in the base step). Then i can be obtained from an n'-th line j ($n' \leq n$) by applying once (case 1) ECQ, DAI, or a replacement rule, or (case 2) any inference or replacement rule. Then (1_S) $j \Vdash_S i$ in case 1 (by Theorem 7) and (1_W) $j \Vdash_W i$ in case 2 (by Theorem 7 and Corollary 3). By the induction hypothesis and the fact that the sequence of the first n' lines of the strong (case 1) or weak (case 2) derivation of i from Γ is a strong (case 1) or weak (case 2) derivation with at most n lines of j from Γ, (2_S) $\Gamma \Vdash_S j$ in case 1, and (2_W) $\Gamma \Vdash_W j$ in case 2. By (1_S), (2_S), and the transitivity of strong semantic validity (which follows from Definition 3), $\Gamma \Vdash_S i$ in case 1. Similarly, by (1_W), (2_W), and the transitivity of weak semantic validity, $\Gamma \Vdash_W i$ in case 2.

PROOF OF COMPLETENESS. Take any pure imperative argument $\langle \Gamma, i \rangle$ and any conjunction i' of all members of Γ. By Theorem 4, there are p, q, p', and q' such that (1) $i \dashv\vdash \ulcorner p \rightarrow\, !q \urcorner$ and (2) $i' \dashv\vdash \ulcorner p' \rightarrow\, !q' \urcorner$. By (1), (2), and Theorem 5: (3) for any m, $m \Vdash\!\!=_S i$ iff $m \vDash \ulcorner p \,\&\, q \urcorner$, $m \Vdash\!\!=_S i'$ iff $m \vDash \ulcorner p' \,\&\, q' \urcorner$, $m \Vdash\!\!=_V i$ iff $m \vDash \ulcorner p \,\&\, {\sim}q \urcorner$, and $m \Vdash\!\!=_V i'$ iff $m \vDash \ulcorner p' \,\&\, {\sim}q' \urcorner$ (see the proof of Corollary 2). *Case 1*: $\Gamma \Vdash_S i$. Then (4) $i' \Vdash_S i$ (by Definition 3). Case 1a: i' is a contradiction. Then, for any r, $i' \Leftrightarrow \ulcorner !(r \,\&\, {\sim}r) \urcorner$ (since i' and $\ulcorner !(r \,\&\, {\sim}r) \urcorner$ are both violated on every m) and thus (by Theorem 5) $i' \dashv\vdash \ulcorner !(r \,\&\, {\sim}r) \urcorner$, so $i' \vdash_S \ulcorner !(r \,\&\, {\sim}r) \urcorner$. Then there is a strong derivation of i from i' (and thus from Γ), since i can be obtained from $\ulcorner !(r \,\&\, {\sim}r) \urcorner$ by ECQ. Case 1b: i' is not a contradiction. Then, by (4) and Theorem 6: (5) for any m, $m \Vdash\!\!=_S i$ only if $m \Vdash\!\!=_S i'$, and (6) for any m, $m \Vdash\!\!=_V i$ only if $m \Vdash\!\!=_V i'$. By (3) and (5): (7) $\ulcorner p \,\&\, q \urcorner \vdash_{CSL} \ulcorner p' \,\&\, q' \urcorner$. By (3) and (6): (8) $\ulcorner p \,\&\, {\sim}q \urcorner \vdash_{CSL} \ulcorner p' \,\&\, {\sim}q' \urcorner$. By using (7), (8), and classical sentential logic, one can show that (9) $p \vdash_{CSL} p'$ and (10) $\ulcorner p \,\&\, (p' \,\&\, q') \urcorner \dashv\vdash_{CSL} \ulcorner p \,\&\, q \urcorner$. To conclude: there is a strong derivation from Γ of i' (by Definition 4), and thus also of $\ulcorner p' \rightarrow\, !q' \urcorner$ (by (2)), and thus also of $\ulcorner p' \rightarrow\, !(p' \,\&\, q') \urcorner$ (by AB), and thus also of $\ulcorner p \rightarrow\, !(p' \,\&\, q') \urcorner$ (by (9) and Theorem 8), and thus also of $\ulcorner p \rightarrow\, !(p \,\&\, (p' \,\&\, q')) \urcorner$ (by AB), and thus also of $\ulcorner p \rightarrow\, !(p \,\&\, q) \urcorner$ (by (10) and DR), and thus also of $\ulcorner p \rightarrow\, !q \urcorner$ (by AB), and thus finally of i (by (1)). *Case 2*: $\Gamma \Vdash_W i$. Then $i' \Vdash_W i$ (by Definition 4 and the observation that any member or conjunction of members of Γ can be obtained from i' by applying replacement rules or ICE or both). Then, by Theorem 6: (11) for any m, $m \Vdash\!\!=_A i'$ only if $m \Vdash\!\!=_A i$, and (12) for any m, $m \Vdash\!\!=_V i$ only if $m \Vdash\!\!=_V i'$. By (3) and (11): (13) $p \vdash_{CSL} p'$. By (3) and (12): (14) $\ulcorner p \,\&\, {\sim}q \urcorner \vdash_{CSL} \ulcorner p' \,\&\, {\sim}q' \urcorner$. By (14) and classical sentential logic: (15) $\ulcorner p \,\&\, q' \urcorner \vdash_{CSL} q$. To conclude: there is a weak derivation from Γ of i' (by Definition 4), and thus also of $\ulcorner p' \rightarrow\, !q' \urcorner$ (by (2)), and thus also of $\ulcorner p \rightarrow\, !q' \urcorner$ (by (13) and Theorem 8), and thus also of $\ulcorner p \rightarrow\, !(p \,\&\, q') \urcorner$ (by AB), and thus also of $\ulcorner p \rightarrow\, !q \urcorner$ (by (15) and Theorem 8), and thus finally of i (by (1)).[5]

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[5] Hansen (2014) provides an alternative sound and complete natural deduction system for SPIL. More precisely, Hansen considers a language of SPIL in which every imperative sentence is either of the form $\ulcorner !q \urcorner$ or of the form $\ulcorner p \rightarrow\, !q \urcorner$ (Hansen uses '$\Rightarrow$' instead of '$\rightarrow$'). This limitation is not crucial: by Theorem 4, every imperative sentence of the language of SPIL is inderderivable with a sentence of the form $\ulcorner p \rightarrow\, !q \urcorner$ by using only replacement rules (which

7. Conclusion

I conclude by noting that in future work I plan to address some of the limitations of SPIL by presenting sound and complete natural deduction systems for three further logics: (1) *First-Order Pure Imperative Logic* (FOPIL), which includes quantifiers and identity but no modal operators; (2) *Sentential Modal Imperative Logic* (SMIL), which includes modal operators but no quantifiers or identity and deals with arguments from declarative or imperative premises (or both) to declarative or imperative conclusions; and (3) *First-Order Modal Imperative Logic* (FOMIL), which combines (1) and (2).

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Hansen does not introduce, although in effect he relies on TA and one of his inference rules corresponds to IC). Hansen's system has six inference rules; five of them correspond to (special cases of) ECQ, IC, Strengthening the Antecedent, and Weakening the Consequent, but the remaining rule is new. (Only the rule that corresponds to a special case of Weakening the Consequent may not be applied in Hansen's "strong deductions", which roughly correspond to strong derivations.) Here is the new rule (which Hansen calls "Contextual Extensionality") in my notation: if $p \vdash_{CSL} \ulcorner q \leftrightarrow r \urcorner$, then $\ulcorner p \rightarrow !q \urcorner$ ► $\ulcorner p \rightarrow !r \urcorner$. Although this rule has no analog in my system, its effects can be simulated by using only replacement rules: if $p \vdash_{CSL} \ulcorner q \leftrightarrow r \urcorner$, then $\ulcorner p \mathbin{\&} q \urcorner \dashv\vdash_{CSL} \ulcorner p \mathbin{\&} r \urcorner$, and then $\ulcorner p \rightarrow !(p \mathbin{\&} q) \urcorner$ and $\ulcorner p \rightarrow !(p \mathbin{\&} r) \urcorner$ are replacement interderivable (by DR), and thus so are also $\ulcorner p \rightarrow !q \urcorner$ and $\ulcorner p \rightarrow !r \urcorner$ (by AB).